Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2023/24

— Exercise Sheet #1 —

Problem 1: Pauli matrices.
Recall the Pauli matrices from the lecture, which in the computational basis $\{|0\rangle, |1\rangle\}$ are of the form

$$X = \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad Y = \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

1. Show that the Pauli matrices are all hermitian, unitary, square to the identity, and different Pauli matrices anticommute.
2. Check the relation $\sigma_\alpha \sigma_\beta = \sum_\gamma i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma + \delta_{\alpha\beta} I$ $(\alpha, \beta, \gamma = 1, 2, 3)$, with $\varepsilon_{\alpha\beta\gamma}$ the fully antisymmetric tensor (i.e. $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$, and zero otherwise).
3. The *trace* $\mathrm{tr}[X]$ is defined as the sum of the diagonal elements of X, i.e., $\mathrm{tr}[X] := \sum_i X_{ii}$. Determine $\mathrm{tr}[I]$, $\mathrm{tr}[\sigma_\alpha]$, and $\mathrm{tr}[\sigma_\alpha \sigma_\beta]$.
4. Write each operator X, Y and Z using bra-ket notation with states from the computational basis.
5. Find the eigenvalues e_i and eigenvectors $|v_i\rangle$ of the Pauli matrices (expressed in the computational basis), and write them in their diagonal form $e_1|v_0\rangle\langle v_0| + e_1|v_1\rangle\langle v_1|$.
6. Write all tensor products of Pauli matrices $\sigma_\alpha \otimes \sigma_\beta$ (including the identity $\sigma_0 = I$) as 4×4 matrices.

Problem 2: Matrix spaces as Hilbert spaces.
Let $\mathcal{V}_d$ be the space of all complex $d \times d$ matrices, and $\mathcal{W}_d \subset \mathcal{V}_d$ the space of all hermitian complex $d \times d$ matrices (i.e. for $M \in \mathcal{W}_d$, $M = M^\dagger$).

1. Show that $\mathcal{V}_d$ forms a vector space over $\mathbb{C}$, and $\mathcal{W}_d$ forms a vector space over $\mathbb{R}$, but not over $\mathbb{C}$. We will in the following always consider $\mathcal{V}_d$ as a complex and $\mathcal{W}_d$ as a real vector space.
2. Show that the Pauli matrices together with the identity, $\Sigma := \{\sigma_i\}_{i=0}^3$, form a basis for both $\mathcal{V}_2$ (over $\mathbb{C}$) and $\mathcal{W}_2$ (over $\mathbb{R}$).
3. Show that
$$(A, B) = \mathrm{tr}[A^\dagger B]$$
defines a scalar product (the "Hilbert-Schmidt scalar product") both for $\mathcal{V}_d$ and for $\mathcal{W}_d$. Here, $\mathrm{tr}[X]$ is the trace, i.e., the sum of the diagonal elements.
4. Show that the Pauli matrices Σ form an orthonormal basis (ONB) with respect to the suitably rescaled Hilbert-Schmidt scalar product.
5. Use the fact that for any scalar product $(\vec{v}, \vec{w})$ and a corresponding ONB $\vec{w}_i$, we can write
$$\vec{v} = \sum_i \vec{w}_i(\vec{w}_i, \vec{v}) ,$$
to express a general matrix in $M \in \mathcal{V}_2$ as
$$M = \sum m_i \sigma_i .$$
What is the form of the m_i? What special property do the m_i satisfy for $M \in \mathcal{W}_2$?
6. Show that a hermitian orthonormal basis also exists for $\mathcal{V}_d$ and $\mathcal{W}_d$. (Ideally, explicitly construct such a basis.)

Problem 3: Eigenectors.

1. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, i.e. $A = A^\dagger$. Show that if $|v\rangle$ is an eigenvector with λ and $|w\rangle$ is an eigenvector with $\mu \neq \lambda$, then $\langle w|v\rangle = 0$.

2. Let $A \in \mathcal{B}(\mathcal{H})$, and let the set $\{\lambda_i\}_{i=1}^n$ be a subset of its eigenvalues. For each $i = 1, \ldots, n$ let $|v_i\rangle$ be an eigenvector. Show that the set $\{|v_i\rangle\}_{i=1}^n$ is linearly independent. *Hint:* consider the polynomial $\frac{(A-\lambda_2 I)(A-\lambda_3 I)\ldots(A-\lambda_n I)}{(\lambda_1-\lambda_2)\ldots(\lambda_n-\lambda_1)}$.

3. Let $A \in \mathcal{B}(\mathcal{H})$, and let $\{\lambda_i\}_{i=1}^n$ be all of its eigenvectors. Show that the spaces $V_i = \mathrm{Ker}((A-\lambda_i)^{m_i})$ are linearly independent for any $m_i \in \mathbb{Z}^+$.

4. Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $AB = BA$. Show that if $|v\rangle$ is an eigenvector of A with eigenvalue λ, then $B|v\rangle$ is either 0 or also an eigenvector with eigenvalue λ.

5. Let $A \in \mathcal{B}(\mathcal{H})$ be such that $AA^\dagger = A^\dagger A$. Let $|v\rangle$ be an eigenvector of A with eigenvalue λ. Show that $|v\rangle$ is an eigenvector of $A^\dagger$ as well with eigenvalue $\bar{\lambda}$.