Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2023/24

— Exercise Sheet #2 —

Problem 4: Measurements.

1. Find a measurement that differentiates between the states $\rho = \frac{1}{2}(\mathbb{1} + aX)$ and $\sigma = \frac{1}{2}(\mathbb{1} + bY)$, for any $a, b \in \mathbb{R}$, $|a| < 1$, $|b| < 1$, $a \neq 0$, $b \neq 0$. (That is, find a measurement that has different outcome probabilities in the two states.)
2. Find a measurement differentiating between $\rho = \frac{1}{2}(\mathbb{1} + aX)$ and $\sigma = \frac{1}{2}(\mathbb{1} + bX)$.
3. Give the outcome probabilities, post measurement states and the expectation value of measuring Y in the state
$$\rho = \frac{1}{2}(\mathbb{1} + aX + bY + cZ).$$

Problem 5: Bloch sphere.

1. Recall that all mixed qubit states are of the form
$$\rho = \frac{1}{2}(\mathbb{1} + aX + bY + cZ),$$
with $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 \leq 1$. Show that the pure states are exactly those where $a^2 + b^2 + c^2 = 1$.
2. Show that one can parametrize any unit vector in two dimensions as
$$|\psi\rangle = e^{i\chi}\big[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\big] \ . \tag{1}$$
Find a, b, c such that
$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + aX + bY + cZ). \tag{2}$$
Where does this point lie on the Bloch sphere?
3. Show that if $\langle\phi|\psi\rangle = 0$, then $|\phi\rangle\langle\phi|$ lies on the exact opposite side of the Bloch sphere.

Problem 6: Ensemble decompositions. In the following we construct different ensemble decompositions of the state
$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \quad p \in [0, 1].$$

1. Check that the above sum is already an ensemble decomposition of ρ.
2. Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Let
$$\sigma = \frac{1}{1-\mu}(\rho - \mu|+\rangle\langle +|).$$
For which values of μ is σ a density matrix? Check that for these values of mu the decomposition $\rho = \mu|+\rangle\langle +| + (1-\mu)\sigma$ is an ensemble decomposition of ρ.
3. For which value of μ is σ a pure state? For this μ give the vector $|\psi\rangle$ such that $\sigma = |\psi\rangle\langle\psi|$.
4. Check that for this value of μ there is a unitary matrix U such that
$$\sqrt{\mu}|+\rangle = U_{00}\sqrt{p}|0\rangle + U_{01}\sqrt{1-p}|1\rangle$$
$$\sqrt{1-\mu}|\psi\rangle = U_{10}\sqrt{p}|0\rangle + U_{11}\sqrt{1-p}|1\rangle.$$
5. Find an ensemble decomposition of ρ with three pure states.