Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2023/24

— Exercise Sheet #3 —

Problem 7: Measurements and filtering

Suppose that a bipartite system AB is initially in the state

$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle \ .$$

The goal of Alice and Bob is to obtain a maximally entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

with some probability by applying local operations only. Specifically, the plan is that Alice will apply a POVM measurement to achieve that.

1. Show that the operators $M_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes I_B$ and $M_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes I_B$, with $0 \leq \gamma \leq 1$, define a POVM measurement. (Note that these describe measurements carried out on Alice's side only!)

2. Determine the outcome probabilities and the post-measurement states for both measurement outcomes.

3. Find a value γ such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.

Problem 8: Trace and partial traces as CPTP maps.

1. Consider the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 9 \\ 5 & 1 & 0 & -1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2),$$

   written in the computational basis, with the basis vectors alphabetically listed (that is, in the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$). Calculate the partial traces and the trace of this matrix.

2. Verify that for all finite dimensional Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, and matrices $X \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$,

$$\mathrm{tr}(X) = (\mathrm{tr}_{\mathcal{H}} \otimes \mathrm{tr}_{\mathcal{K}})(X).$$

3. Show that the trace and the partial traces are all trace preserving (the trace of a complex number is itself).

4. Show that the trace and the partial traces all admit a Kraus representation, and thus they are completely positive.

Problem 9: CPTP maps.

In this problem, we will study some commonly appearing CPTP maps (quantum channels). In addition to the problems listed, verify for each map that it is CPTP (completely positive trace preserving) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z \ .$$

Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z) \ ,$$

i.e., it acts as

$$\rho = \tfrac{1}{2}(I + r_x \cdot X + r_y \cdot Y + z_z \cdot Z) \mapsto \tfrac{1}{2}(I + (1-2p)r_x \cdot X + (1-2p)r_y \cdot Y + r_z \cdot Z).$$

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$M_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1| \ ,$$

where $0 \leq \gamma \leq 1$. Here, M_0 describes a decay from $|1\rangle$ to $|0\rangle$, and γ corresponds to the decay rate.

(a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

where $0 \leq p \leq 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

(b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter γ_1 and consider another amplitude damping channel $\mathcal{E}_2$ with parameter γ_2. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1-(1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the γ's as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z \ .$$

Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}I$.

Hint: Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.