Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2023/24

— Exercise Sheet #4 —

Problem 10: Decay of entanglement.

Consider the maximally entangled state $\rho = |\Omega\rangle\langle\Omega|$, where $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like ρ are typically not stable, but decay over time. A typical evolution is that the off-diagonal elements decay relatively quickly to zero with a timescale T_2 ("dephasing"), while the diagonal elements become equal with a longer timescale T_1 ("decoherence"). Such an evolution is described as

$$\rho(t) = p_+|00\rangle\langle 00| + p_-|01\rangle\langle 01| + p_-|10\rangle\langle 10| + p_+|11\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|00\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|11\rangle\langle 00| \ ,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.

1. Give the matrix form of $\rho(t)$.
2. Determine the values of T_1 and T_2 for which $\rho(t) \geq 0$ for all times t. (You should find that T_2 cannot be much larger than T_1, otherwise $\rho(t)$ becomes unphysical – that is, there is indeed a natural reason why we would typically expect dephasing to occur on the faster timescale.)
3. What is the limit $\lim_{t\to\infty} \rho(t)$? Is it entangled?
4. Take the partial transpose $\rho(t)^{T_B} = (\mathbb{I} \otimes T)(\rho(t))$ and give its matrix form.
5. Calculate the eigenvalues of $\rho(t)^{T_B}$.
6. Sketch how the eigenvalues change over time for $T_1 = T_2 = 1$. What it the asymptotic limit?
7. Find the time t_{sep} after which $\rho(t_{\text{sep}})$ becomes separable.

Problem 11: Entangelment witnesses.

Consider a bipartite system with $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$. Let $W := \mathbb{I} - d|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i, i\rangle$.

1. Show that $\mathrm{tr}[W\rho] \geq 0$ for separable states ρ. That is, $\mathrm{tr}[W\rho] \leq 0$ implies that ρ is entangled. Such an operator W is called an entanglement witness.
2. Consider the family
$$\rho_{\text{iso}}(\lambda) = \lambda \frac{\mathbb{I}}{d^2} + (1-\lambda)|\Omega\rangle\langle\Omega|$$
of *isotropic states*. In which range of λ is $\rho_{\text{iso}}(\lambda) \geq 0$? In which range of λ does W detect that $\rho_{\text{iso}}(\lambda)$ is entangled?
3. Consider the case $d = 2$. Does W detect the antisymmetric state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ as entangled? Generally, which property must a pure state satisfy to be detected by W?
4. Verify that $d \cdot (\Lambda \otimes \mathbb{I})(|\Omega\rangle\langle\Omega|) = W$ for the map Λ defined by $\Lambda(\rho) := d\,\mathrm{tr}_B\big[W^T(\mathbb{I}_A \otimes \rho_B^T)\big]$. (That is, Λ maps to $\frac{1}{d}W$ through the Choi-Jamiolkowski isomorphism). What is Λ? Show that it is positive but not completely positive.
5. For a two-qubit system, in which range of λ does Λ detect that $\rho_{\text{iso}}(\lambda)$ is entangled? Does Λ detect the antisymmetric state?

Problem 12: Manipulation of entangled states.

1. Show that the maximally entangled state $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$ has the property

$$(\mathbb{I} \otimes O)|\Omega\rangle = (O^T \otimes \mathbb{I})|\Omega\rangle$$

   for every $O \in \mathcal{B}(\mathcal{H})$.

2. Show that if Alice and Bob share a maximally entangled state and Alice applies a CP map T (e.g. time evolution) while Bob doing nothing, then the result is the same as if Bob applied another CP map S on the state while Alice doing nothing:

$$(T \otimes \mathbb{I})(|\Omega\rangle\langle\Omega|) = (\mathbb{I} \otimes S)(|\Omega\rangle\langle\Omega|).$$

   How does T and S relate to each other? Assume that T is in addition trace preserving. What extra property does S have? Is it tace preserving?

3. Let Alice and Bob share a maximally entangled state and assume that Alice perform a POVM measurement described by measurement operators $\{M_i\}_{i \in I}$, and gets outcome i. Show that the corresponding post-measurement state can be obtained with the same probability if instead of Alice, Bob performs a suitable measurement on the maximally entangled state. Construct such a measurement.