Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2023/24

— Exercise Sheet #5 —

Problem 13: Entanglement witness for a given state.

In this problem we will construct an entanglement that detects the entanglement of a given pure state.

1. Show that any separable state can be written as the convex combination of pure separable states (i.e. pure states of the form $|\phi\rangle \otimes |\chi\rangle$).
2. Let $|\Psi\rangle$ be a pure entangled state. Show that there is a separable state ρ_0 such that for any other separable state ρ, $\langle\Psi|\rho|\Psi\rangle \leq \langle\Psi|\rho_0|\Psi\rangle$.
3. Let $W = \langle\Psi|\rho_0|\Psi\rangle \cdot \mathbb{I} - |\Psi\rangle\langle\Psi|$. Show that $\mathrm{tr}\{\rho W\} \geq 0$ for all separable ρ.
4. Show that W is an entanglement witness that detects the entanglement of $|\Psi\rangle$.

Problem 14: Majorization

Let $x, y \in \mathbb{R}^n_{\geq 0}$. Assume that $x_1 \geq x_2 \geq \dots x_n$ and $y_1 \geq y_2 \geq \dots y_n$. We say that y majorizes x, and write $x \prec y$, if for all $k = 1, 2, \dots, n$,

$$x_1 + \cdots + x_k \leq y_1 + \cdots + y_k.$$

In this problem, we prove that $x \prec y$ implies that $x = \sum_j q_j P_j y$ for some probability distribution q_j and permutation matrices P_j, where $x, y \in \mathbb{R}^d_{\geq 0}$. The proof will proceed by induction in the dimension d of the space.

1. Show that there exist k and $t \in [0, 1]$ such that $x_1 = ty_1 + (1 - t)y_k$. For which k does this work? For the following steps, we choose the *smallest such* k.
2. Define $D = tI + (1 - t)T$, where T is the permutation matrix which transposes the 1st and k-th matrix elements. What are the components of the vector Dy?
3. Define x' and y' by eliminating the first entry from x and Dy, respectively. Show that $x' \prec y'$.
4. Show that this way, we can inductively prove the claim.

Problem 15: Fidelity.

1. Prove that for normalized vectors $|\psi\rangle$ and $|\phi\rangle$,
$$\big|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle\big| \leq \sqrt{8}\,\sqrt{1 - |\langle\psi|\phi\rangle|}\,\|O\|_\infty\ ,$$
with $\|O\|_\infty = \|O\|_{\mathrm{op}} = \sup_{|\psi\rangle} \frac{\|O|\psi\rangle\|}{\||\psi\rangle\|}$. Use this to prove
$$\big|\langle\psi|O|\psi\rangle - \langle\phi|O|\phi\rangle\big| \leq 2\sqrt{\delta}\|O\|_\infty\ , \qquad (*)$$
to leading order in δ, where $\delta = 1 - F$, with $F = |\langle\psi|\phi\rangle|^2$ the fidelity.
2. Use the operator Hölder inequality
$$|\mathrm{tr}(AB)| \leq \|A\|_1 \|B\|_\infty\ ,$$
where the *trace norm* $\|A\|_1$ is the sum of the singular values of A (i.e. for hermitian A the sum of the absolute value of the eigenvalues) to prove $(*)$ directly (and without the need for a leading-order approximation).
(Of course, any alternative proof is also fine.)