Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2023/24

— Exercise Sheet #10 —

Problem 26: Quantum Error Correction Conditions

Check the Quantum Error Correction Conditions

$$\langle\hat{\imath}|E_\alpha^\dagger E_\beta|\hat{\jmath}\rangle = c_{\alpha\beta}\delta_{ij} \tag{1}$$

1. for the 3-qubit bit flip code, and the error model with operators

   $$E_0 = \sqrt{(1-p)^3}(I\otimes I\otimes I)\ ,$$
   $$E_1 = \sqrt{(1-p)^2p}(I\otimes I\otimes X)\ ,\ E_2 = \sqrt{(1-p)^2p}(I\otimes X\otimes I)\ ,\ E_3 = \sqrt{(1-p)^2p}(X\otimes I\otimes I)\ .$$

   (Observe that the individual normalization of the E_α is not important for the correctness of Eq. (1), it only affects the specific $c_{\alpha\beta}$.)

2. (if you feel enthusiastic) for the 9-qubit code and arbitrary single-qubit errors, i.e., where $E_0 \propto I$, and $E_1, \ldots, E_{27}$ are a single Pauli operator in any one position.
   (This is not as tedious as it sounds, as you can use the structure of the code. One possibility is to work with the basis dual to the one given in the lecture, i.e. the sum and difference of the two basis states, and compare these two states $|\hat{+}\rangle$ and $|\hat{-}\rangle$ the computational basis – this quite directly gives the δ_{ij}, and it only remains to show that the prefactor is independent of i. But there are certainly other ways to use the structure.)

Problem 27: Working with the 3-qubit bit flip code

In this problem, we will study how to work with the 3-qubit bit flip code, i.e., how to explicitly perform the error correction, and also look at how to implement some gates without decoding the information.

1. Consider a qubit encoded with the 3-qubit code. Find a circuit which measures the error syndrome (i.e. which of the three qubits, if any, differs from the others), consisting of elementary gates and single-qubit measurements in the computational basis, and possibly using ancillas in the $|0\rangle$ state. (You should only need CNOT gates.) For each measurement outcome, give the correction operation.

2. Show that instead of measuring the ancillas, we can also perform quantum gates for the correction, and then discard (trace out) the ancillas, without the need for a measurement. Can this also be done only with CNOTs and simple single-qubit gates (Hadamard, Pauli)?

3. Show that the Pauli operators on the encoded (logical) qubit can be implemented by acting with single-qubit gates on the physical qubits, without decoding the code. (Again, single-qubit Paulis should suffice.)

4. Now consider two qubits, each encoded with a 3-qubit code. What happens when we apply CNOT gates between all three pairs of physical qubit (i.e. between qubit 1 of the 1st qubit and qubit 1 of the 2nd qubit, etc.)? (Logical gates which can be implemented in this way are called *transversal gates*; note that the same property also holds for the Paulis above.)