3. Linear algebra

The language of quantum mechanics is based on Hilbert spaces. In this course we only think of quantum objects with finite degree of freedom eg. polarization of photon (circular: left/right, linear: horizontal/vertical), not infinite degrees of freedom (such as position in space). This simplifies the language, and we only need to learn about finite dimensional Hilbert spaces (over the complex numbers). These objects are f.d. vector spaces with a scalar product.

3.1 Notations

- Complex unit: i.

$$\mathbb{C}^d = \left\{ \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} \middle| \, a_0, \dots, a_{d-1} \in \mathbb{C} \right\} \text{ for a fixed } d.$$

- You can then multiply w/ scalar, e.g.

$$i \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ i \end{pmatrix}, \quad 2 \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- And add up such objects:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ i \end{pmatrix} = \begin{pmatrix} 3 \\ 1+i \end{pmatrix}$$

When referring to vectors, we surround the variable with a vertical line and an angle bracket:

$$|v\rangle = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \in \mathbb{C}^2.$$

We read "ket-v". Given $|v\rangle, |w\rangle \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$, we also write

$$\lambda |v\rangle = |\lambda v\rangle \text{ and } |v\rangle + |w\rangle = |v+w\rangle.$$

For example, if $|v\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $|w\rangle = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, then

$$|v+2w\rangle = |v\rangle + 2|w\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

A <u>basis</u> of $\mathbb{C}^d$ is the collection of vectors

$$|v_0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ |v_1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix},\ \dots,\ |v_{d-1}\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

This basis we refer to as the standard/computational basis and write $|0\rangle, |1\rangle, \dots, |d-1\rangle$ instead of $|v_0\rangle, |v_1\rangle, \dots, |v_{d-1}\rangle$.

⚠ In this notation

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = |0\rangle + 2\cdot|1\rangle \neq |0+2\cdot 1\rangle.$$

⚠ $$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In general, one can use any symbol in "ket".

<u>Eg</u>: in $\mathbb{C}^2$ sometimes we write $|\uparrow\rangle$ and $|\downarrow\rangle$ instead of $|0\rangle$ and $|1\rangle$.

Def: A finite dimensional Hilbert space is

- A finite dimensional vector space $\mathcal{H}$
- Equipped with a scalar product, that is:
  - A map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, $(a,b) \mapsto \langle a|b\rangle$
  - S.t. it is anti-symmetric:

    $\langle b,a\rangle = \overline{\langle a,b\rangle} \quad \forall |a\rangle, |b\rangle \in \mathcal{H}$
  - Linear in the 2^{nd}, and conj. linear in 1^{st} variable
    - $\langle a+\lambda b, c\rangle = \langle a,c\rangle + \bar{\lambda}\langle b,c\rangle$
    - $\langle a, b+\lambda c\rangle = \langle a,b\rangle + \lambda\langle a,c\rangle$

    $\forall a,b,c \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.
  - Positive definite:

    $\langle a,a\rangle \geq 0 \ \forall |a\rangle \in \mathcal{H}$ and $\langle a,a\rangle = 0 \Leftrightarrow |a\rangle = 0$.

In $\mathbb{C}^d$ the "standard" scalar product is given by the following recipe:

$$|v\rangle = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix} \quad \text{and} \quad |w\rangle = \begin{pmatrix} w_0 \\ \vdots \\ w_{d-1} \end{pmatrix}$$

$$\Rightarrow \langle v|w\rangle = \bar{v}_0 w_0 + \bar{v}_1 w_1 + \dots + \bar{v}_{d-1} w_{d-1}.$$

Eg. $|v\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $w = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \Rightarrow \langle v|w\rangle = (1 \ -i)\begin{pmatrix} 1+i \\ 1 \end{pmatrix} = 1.$

A convenient way to remember:

$$\langle v_0 | w_0 \rangle = (\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{d-1}) \begin{pmatrix} w_0 \\ \vdots \\ w_{d-1} \end{pmatrix}.$$

For example, $|v\rangle = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$, $|w\rangle = \begin{pmatrix} i \\ 2 \end{pmatrix}$, then

$$\langle v|w\rangle = (1, 1-i)\begin{pmatrix} i \\ 2 \end{pmatrix} = i + 2(1-i) = 2-i.$$

$$\langle w|v\rangle = (-i, 2)\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = -i + 2(1+i) = 2+i = \overline{(2-i)},$$

The basis vectors $|0\rangle, |1\rangle, \dots, |d-1\rangle$ satisfy

$$\langle j|k\rangle = (0 \dots \underset{\substack{\uparrow \\ j}}{1} \dots 0)\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k = 0 \text{ if } j \neq k$$

We say that this is an orthogonal basis.

The row vecter $(\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{d-1})$ is denoted as $\langle v|$, read as "bra-v" and it refers to the linear functional $|w\rangle \mapsto \langle v|w\rangle$.

For example, $\langle 0| = (1, 0, \dots, 0)$, $\langle 1| = (0, 1, 0, \dots, 0)$; $\dots$ $\langle d-1| = (0, \dots, 0, 1)$ is a basis of the row vecters.

Remark: careful with the conjugation!

$$|v\rangle = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = 1 \cdot |0\rangle + (1+i)\,|1\rangle.$$

$$\langle v| = (1, 1-i) = 1 \cdot \langle 0| + \underbrace{(1-i)}_{\text{this is } \overline{1+i}.} \langle 1|$$

...

Let us consider $|v\rangle \in \mathcal{H}$,

$$v = \begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix} = v_0|0\rangle + v_1|1\rangle + \ldots + v_{d-1}|d-1\rangle.$$

The norm of $|v\rangle$ is

$$\|v\| = \left(\langle v|v\rangle\right)^{1/2} = \left(\sum_i |v_i|^2\right)^{1/2}.$$

Remember: $\langle v|v\rangle \geq 0 \;\; \forall |v\rangle \in \mathcal{H}$.

For example, if $|v\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\|v\|^2 = (1\,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \Rightarrow \||0\rangle\| = 1.$$

In general,

$$\langle j|k\rangle = \begin{cases} 0 \text{ if } j \neq k \\ 1 \text{ if } j = k \end{cases} =: \delta_{jk}.$$

Matrices (linear maps) in bra-ket notation

Reminder: bra-ket notation $\langle \cdot | \cdot \rangle = (\ldots)\begin{pmatrix} \vdots \end{pmatrix}$

Eg.: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$, $(1\ 0) = \langle 0|$, $(0\ 1) = \langle 1|$,

and in general $\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = |v\rangle = v_0|0\rangle + v_1|1\rangle$.

Today: matrices:

$$M = \begin{pmatrix} M_{00} & M_{01} & \cdots & M_{0,d-1} \\ \vdots & & & \\ M_{s-1,0} & M_{s-1,1} & \cdots & M_{s-1,d-1} \end{pmatrix} \quad s \times d \text{ matrix}$$

Such a matrix defines $\mathbb{C}^d \to \mathbb{C}^s$ lin. map:

$$\underset{\mathbb{C}^d}{\underset{\uparrow}{M|v\rangle}} = \begin{pmatrix} M_{00} & M_{01} & \cdots & M_{0,d-1} \\ \vdots & & & \vdots \\ M_{s-1,0} & \cdots & & M_{s-1,d-1} \end{pmatrix}\begin{pmatrix} v_0 \\ \vdots \\ v_{d-1} \end{pmatrix} = \underset{\mathbb{C}^s}{\underset{\uparrow}{\underbrace{\begin{pmatrix} \sum_k M_{0k} v_k \\ \vdots \\ \sum_k M_{s-1,k} v_k \end{pmatrix}}}}$$

Eg. $\boxed{1}$ $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ i \\ 3 \end{pmatrix} = \begin{pmatrix} 6+i \\ 12+4i \end{pmatrix}$.

$\boxed{2}$ $\underbrace{(2i\ 3\ 5)}_{1\times 3 \text{ mx}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2i+5$ — $\langle v|\cdot|w\rangle = \langle v|w\rangle$

Row vectors can be viewed as lin. maps $\mathbb{C}^d \to \mathbb{C}$. They are called lin. functionals. The row vector $\langle v|$ defines the lin. fcn. $\langle v| \mapsto \langle v|w\rangle$.

⚠ $\langle v| = (a \ b \, .. \ x) \iff |v\rangle = \begin{pmatrix} \bar{a} \\ \bar{b} \\ \vdots \\ \bar{x} \end{pmatrix}$

Important matrices:

$$\left.\begin{aligned} X &= \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ Y &= \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ Z &= \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\} \text{Pauli matrices}$$

$$I = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix multiplication

Two matrices $d \times s$ & $s \times r$ can be multiplied; result: $d \times r$ matrix

$$\begin{pmatrix} M_{00} & \cdots & M_{0,s-1} \\ \vdots & & \\ M_{d-1,0} & \cdots & M_{d-1,s-1} \end{pmatrix} \begin{pmatrix} N_{00} & \cdots & N_{0,r-1} \\ \vdots & & \vdots \\ N_{s-1,0} & \cdots & N_{s-1,r-1} \end{pmatrix} =$$

$$MN = \begin{pmatrix} \sum_k M_{0k} N_{k0} & \cdots & \sum_k M_{0k} N_{k,r-1} \\ \vdots & & \vdots \\ \sum_k M_{d-1,k} N_{k,0} & & \sum_k M_{d-1,k} N_{k,r-1} \end{pmatrix}$$

$(MN)_{ij} = \sum_k$

<u>Example:</u> 2×2 mx $\cdot$ 2×1 mx (column vector) $\Rightarrow$ 2×1 mx

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

Or 2×2 mx $\cdot$ 2×2 mx $\Rightarrow$ 2×2 mx

$$X\cdot Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\mathbb{1}$$

$$Y\cdot X = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\mathbb{1}.$$

HW: all products!

In particular, $XY = -YX \neq YX$.

Mx multiplication is <u>not</u> commutative!

In general, even if you can do $A\cdot B$, $B\cdot A$ is not guaranteed to exist.

⚠ I always work with column vectors $\Rightarrow$ $A\cdot B$ means: B acts first, A second.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ is called identity and it has the property

$$IM = MI = M \quad \forall\ 2\times 2 \text{ matrix}.$$

This works in higher dimensions, I will write I or $\mathbb{1}$ or 1 for identity mx in all dimensions (unless there might be a confusion of dimension).

So depending on context, I is either 2×2 or 3×3 etc.

Another example: 2x1 mx · 1x2 mx

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

These matrices form a basis of 2x2 matrices:

$$\begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix} = \begin{pmatrix} M_{00} & 0 \\ 0 & 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 & 0 \\ 0 & M_{11} \end{pmatrix} =$$

$$= M_{00}\,|0\rangle\langle 0| + M_{01}\,|0\rangle\langle 1| + M_{10}\,|1\rangle\langle 0| + M_{11}\,|1\rangle\langle 1| .$$

In general also works,

$$M = \sum_{jk} M_{jk}\,|j\rangle\langle k| .$$

Example:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0| .$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0| .$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1| .$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1| .$$

Calculating in this form:

$$M|\ell\rangle = \sum_j M_{jk}\, |j\rangle \underbrace{\langle k| \cdot |\ell\rangle} = \sum_j M_{j\ell} |j\rangle .$$

$$\langle k|\ell\rangle = \begin{cases} 0 \text{ if } k \neq \ell \\ 1 \text{ if } k = \ell \end{cases}$$

I will write δ_{jk} for this: 1 if $j = k$, and 0 else. I call it Kronecker delta. With this,

$$I = \sum_j |j\rangle\langle j| = \sum_{jk} \delta_{jk} |j\rangle\langle k| .$$

Notice now that

$$\langle j|M|k\rangle = \langle j| \sum_{\ell m} M_{\ell m} |\ell\rangle\langle m| \cdot |k\rangle$$

$$= \sum_{\ell m} M_{\ell m} \underbrace{\langle j|\ell\rangle}_{\delta_{j\ell}} \underbrace{\langle m|k\rangle}_{\delta_{mk}} = M_{jk} .$$

Adjoint

Given an $r \times s$ mx

$$M = \begin{pmatrix} M_{00} & \cdots & M_{0,s-1} \\ M_{r-1,0} & \cdots & M_{r-1,s-1} \end{pmatrix}$$

Its adjoint, $M^\dagger$ is an $s \times r$ mx def. as the conjugate transpose:

$$M^\dagger = \begin{pmatrix} \overline{M}_{00} & & \overline{M}_{r-1,0} \\ & \ddots & \\ \overline{M}_{0,s-1} & & \overline{M}_{s-1,r-1} \end{pmatrix} \quad \text{i.e. } (M^\dagger)_{ij} = \overline{M}_{ji}.$$

Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In bra-ket notation,

$$(|0\rangle\langle 1|)^\dagger = |1\rangle\langle 0|.$$

$$Y^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y.$$

Notice that $\dagger$ is an anti-linear op:

$$(\mu M + \nu N)^\dagger = \overline{\mu} M^\dagger + \overline{\nu} N^\dagger.$$

Also $(MN)^\dagger = N^\dagger M^\dagger.$

In bra-ket notation, we can write

$$M^+ = \left(\sum_{jk} M_{jk} |j\rangle\langle k| \right)^+ =$$

$$= \sum_{jk} \overline{M}_{jk} |k\rangle\langle j| = \sum_{jk} \overline{M}_{kj} |j\rangle\langle k|,$$

As

$$M^+ = \sum_{jk} (M^+)_{jk} |j\rangle\langle k| \quad \text{as well,}$$

we have, as expected,

$$(M^+)_{jk} = \overline{M}_{kj}.$$

This can actualy be used for abstract definition:

$$\langle j | M^+ | k \rangle = \overline{\langle k | M | j \rangle},$$

and thus for any $|v\rangle$, $|w\rangle$

$$\langle v | M^+ w \rangle = \langle v | M^+ | w \rangle = \overline{\langle w | M | v \rangle} =$$

$$= \overline{\langle w | M v \rangle} = \langle M v | w \rangle,$$

Here note that $\langle v|$ is conj. transpose of $|v\rangle$. Can be used for abstract def:

$$\langle w | M v\rangle = \langle M^{+} w | v\rangle \quad \forall\, v, w.$$

<u>Def</u> Let M be a $d \times d$ matrix, $M \in \mathcal{M}_d$.

We say that M is <u>self-adjoint</u> or Hermitian if

$$M^{+} = M.$$

<u>Ex:</u> X, Y, Z, I are self-adjoint.

Def We say that U is unitary if

$$UU^{+} = U^{+}U = \mathbb{1}.$$

Remark: in finite dimensions both
$UU^+ = 1$ and $U^+U = \mathbb{1}$ implies already
that U is unitary. (U is a square matrix)

<u>Example</u>:

$$X^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$Y^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & \bar{i} \\ \overline{-i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

$$Z^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z.$$

As

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

and they are self-adjoint, they are unitaries:

$$XX^+ = YY^+ = ZZ^+ = \mathbb{1}.$$

Norm of linear maps:

Let $M: \mathcal{H}_1 \to \mathcal{H}_2$ be linear.

Then the norm of M is

$$\|M\| = \sup_{\substack{v \in \mathcal{H}_1 \\ v \neq 0}} \frac{\|Mv\|}{\|v\|} .$$

Eigenvalues, eigenvectors

Let $M \in \mathrm{End}(\mathcal{H})$. If

$M|v\rangle = \lambda |v\rangle$ for some $\lambda \in \mathbb{C}$, $|v\rangle \in \mathcal{H}$, $|v\rangle \neq 0$,

Then

- v is called an eigen vector of M.
- λ is called an eigenvalue of M.

The set of eigenvalues is called the spectrum of M.

How to find them:

$|v\rangle \in \mathrm{Ker}(M - \lambda \mathbb{1})$

So solutions of

$\det(M - \lambda \mathbb{1}) = 0$ are the spectrum.

Eig. vect. corr. to diff. eig. vecter are lin. indep.
$\#$ of eig. vecters $\leq \dim(\mathcal{H})$.

$=$: Diagonalizable, $M = X^{-1} D X$ w/ D diagonal

$<$: Not diagonalizable.

Eg. $\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a unique eig. value : 0, and for it, unique eig. vecter $|1\rangle$.

Sufficient condition for diagonalizability:

$[M, M^\dagger] = 0$.

For example, all Hermitian and all unitary matrices are diagonalizable.

Hermitian matrices are diagonalizable by a unitary:

$$M = \mathcal{U} D \mathcal{U}^\dagger = \mathcal{U} \sum_k \lambda_k |k\rangle\langle k| \mathcal{U}^\dagger =$$

$$= \sum_k \lambda_k |v_k\rangle\langle v_k|$$

$$= \sum_{\lambda \in \mathrm{Spec}(M)} \lambda P_\lambda = \sum_{\lambda \in \mathrm{Spec}(M)} \lambda \sum_k |v_{k,\lambda}\rangle\langle v_{k,\lambda}|$$

Eg:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Char. poly: } \lambda^2 - 1 = 0 \to \text{Spec} = \{+1, -1\}$$

Eig. vectors: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ for $+1$.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$ for -1.

$$X = |+\rangle\langle +| - |-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ 1) -$$

$$-\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1\ -1) =$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Y, Z: similar, same spectrum!

Def: We say that $M:\mathcal{H}\to\mathcal{H}$ is positive semidefinite if $\forall\, |v\rangle \in \mathcal{H}$

$$\langle v|M|v\rangle \geq 0.$$

We write $M\geq 0$ if M is positive semidefinite.

We say that M is positive definite if $\forall\, |v\rangle\in\mathcal{H}$, $|v\rangle\neq 0$

$$\langle v|M|v\rangle > 0.$$

We write $M>0$ if M is positive definite.

I will say positive instead of pos. semidefinite.

For example,

- $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is positive (def.), as

$$\langle v|v\rangle \geq 0 \text{ and } \langle v|v\rangle=0 \Rightarrow |v\rangle=0.$$

- $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ is <u>not</u> positive as

$$\langle 1|Z|1\rangle = (0\ 1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -1 < 0.$$

- Diag. mx ≥ 0 iff $\forall$ entry ≥ 0.

Thm: Let $M: \mathcal{H} \to \mathcal{H}$ be linear. The following are equivalent:

(1) M is positive: $\langle v|M|v\rangle \geq 0 \ \forall v \in \mathcal{H}$

(2) M is Hermitian and all of its eig. values are ≥ 0

(3) $\exists X: \mathcal{H} \to \mathcal{H}$ s.t. $M = X^{\dagger}X$.

Proof: (1) $\Rightarrow$ (2) Let $|v\rangle, |w\rangle \in \mathcal{H}$, then

$$\langle v+w|M|v+w\rangle - \langle v|M|v\rangle - \langle w|M|w\rangle =$$

$$= \langle v|M|w\rangle + \langle w|M|v\rangle \in \mathbb{R}$$

$$\langle v+iw|M|v+iw\rangle - \langle v|M|v\rangle - \langle w|M|w\rangle =$$

$$= i\langle v|M|w\rangle - i\langle w|M|v\rangle \in \mathbb{R},$$

or

$$\langle v|M|w\rangle - \langle w|M|v\rangle \in i\mathbb{R}.$$

This means that

$$\langle w|M|v\rangle = \overline{\langle v|M|w\rangle} = \overline{\langle M^{\dagger}v|w\rangle} = \langle w|M^{\dagger}|v\rangle.$$

As it holds $\forall\ |v\rangle, |w\rangle \in \mathcal{H}$, $M = M^{\dagger}$, i.e. M is self-adjoint.

Let now $|v\rangle$ be an eig. vector w/ eig. value λ, then

$$\langle v|Mv\rangle = \langle v|\lambda v\rangle = \lambda\langle v|v\rangle,$$

and thus, as $\langle v|v\rangle > 0$, $\lambda > 0$ as well.

(2) $\Rightarrow$ (3) We can write

$$M = \mathcal{U} D \mathcal{U}^\dagger \quad \text{w/ } D \text{ diag}, \ \mathcal{U} \text{ unitary.}$$

$D \geq 0 \Leftrightarrow \forall i \ D_i \geq 0$. We can then define $D^{1/2} = \sum_i \sqrt{D_i}\ |i\rangle\langle i| \geq 0$.

$$M = \mathcal{U} D^{1/2} D^{1/2} \mathcal{U}^\dagger = X^\dagger X \quad \text{w/ } X = D^{1/2}\mathcal{U}^\dagger.$$

(3) $\Rightarrow$ (1)

$$\langle v|X^\dagger X|v\rangle = \langle Xv|Xv\rangle \geq 0.$$

$\square$

The proof was not part of the lecture, but it is good to know.

The set of self-adjoint matrices form a <u>real</u> vector space:

- $A = A^+,\ B = B^+ \Rightarrow (A+B)^+ = A^+ + B^+ = A+B$
- $A = A^+,\ \lambda \in \mathbb{R} \Rightarrow (\lambda A)^+ = \lambda A^+ = \lambda A,$

But for example

$$(iA)^+ = \bar{i} A^+ = -i A^+ = -iA \neq iA.$$

On this set there is a partial order:

$$A \leq B \iff 0 \leq B - A.$$

<u>Exercise #1</u> show that if $A \leq B$, then

- $XAX^+ \leq XBX^+$ for any (even non-square) X.

<u>Exercise #2</u>: find example s.t.

$$A \leq B, \text{ but } CA \nleq CB.$$

Tensor product of vector spaces:

Def: Let $\mathcal{H}_A, \mathcal{H}_B$ be (finite dim.) Hilbert spaces, with ONB $\{|i\rangle_A\}_{i=0}^{d_A-1}$ and $\{|j\rangle_B\}_{j=0}^{d_B-1}$.

Their tensor product is another Hilb. space with dim. $d_A \cdot d_B$ and ONB

$$\left\{ |i\rangle_A \otimes |j\rangle_B \,\middle|\, i = 0..d_A-1, j = 0..d_B-1 \right\}.$$

I will also write $|ij\rangle$ for $|i\rangle \otimes |j\rangle$.

Any vector from $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form:

$$|v\rangle = \sum_{ij} v_{ij} \, |ij\rangle$$

Scalar product of two generic vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\langle v|w\rangle = \sum_{ijkl} \overline{v_{ij}} \, w_{kl} \langle ij|kl\rangle$$

$$= \sum_{ijkl} \overline{v_{ij}} \, w_{kl} \, \delta_{ik} \delta_{jl} =$$

$$= \sum_{ij} \overline{v_{ij}} \, w_{ij}.$$

We can form tensor products of vectors:
Given $|v\rangle \in \mathcal{H}_A$, $|v\rangle = \sum_j v_j |j\rangle$ and $|w\rangle \in \mathcal{H}_B$, $|w\rangle = \sum_j w_j |j\rangle$, their tensor product is a vector $|v\rangle \otimes |w\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ given by

$$|v\rangle \otimes |w\rangle = \sum_{ij} v_i w_j \, |ij\rangle .$$

The tensor product is distributive:

- $|\lambda_1 v_1 + \lambda_2 v_2\rangle \otimes |w\rangle = \lambda_1 |v_1\rangle \otimes |w\rangle + \lambda_2 |v_2\rangle \otimes |w\rangle$
- $|v\rangle \otimes |\mu_1 w_1 + \mu_2 w_2\rangle = \mu_1 \cdot |v\rangle \otimes |w_1\rangle + \mu_2 |v\rangle \otimes |w_2\rangle$

We can define tensor product basis-independently, then these are the defining relations.

<u>Example</u>: $\mathcal{H}_A = \mathbb{C}^2$ w/ basis $\{|0\rangle_A, |1\rangle_A\}$,

$\mathcal{H}_B = \mathbb{C}^2$ w/ basis $\{|0\rangle_B, |1\rangle_B\}$.

Here the subscript A/B serve simply to distinguish between vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$, not strictly necessary.

Then $\mathcal{H}_A \otimes \mathcal{H}_B$ has basis

$$|0\rangle_A \otimes |0\rangle_B = |00\rangle \qquad |1\rangle_A \otimes |0\rangle_B = |10\rangle$$
$$|0\rangle_A \otimes |1\rangle_B = |01\rangle \qquad |1\rangle_A \otimes |1\rangle_B = |11\rangle.$$

$\mathcal{H}_A \otimes \mathcal{H}_B$ is just a 4-dimensional vector space, elements are column vectors. We order the basis alphabetically:

$$|v\rangle = 3|00\rangle + 2i|01\rangle + i|10\rangle - |11\rangle = \begin{pmatrix} 3 \\ 2i \\ i \\ -1 \end{pmatrix} \begin{array}{l} \leftarrow |00\rangle \\ \leftarrow |01\rangle \\ \leftarrow |10\rangle \\ \leftarrow |11\rangle \end{array}.$$

Given $|v\rangle = |0\rangle + |1\rangle$ and $|w\rangle = |0\rangle - |1\rangle$,

$$|v\rangle \otimes |w\rangle = (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) = |00\rangle - |01\rangle + |10\rangle - |11\rangle = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} v_0 \cdot |w\rangle \\ v_1 |w\rangle \end{pmatrix}.$$

Note: we have already seen a tensor product space: the set of matrices.
Given $\mathcal{H}$ w/ basis $\{|j\rangle\}_{j=0}^{d-1}$, the set of row vectors, $\mathcal{H}^\dagger$, is a vector space w/ basis $\{\langle j|\}_{j=0}^{d-1}$. Their tensor product $\mathcal{H}\otimes\mathcal{H}^\dagger$ has basis $\{|j\rangle\langle k|\}_{j,k=0}^{d-1}$;
thus $\mathcal{H}\otimes\mathcal{H}^\dagger \cong B(\mathcal{H})$, the set of matrices.

Note: Not every vector in $\mathcal{H}_A\otimes\mathcal{H}_B$ is a tensor product of two vectors.
For example,

$$|00\rangle + |11\rangle \neq |v\rangle\otimes|w\rangle.$$

A convenient way to see this: form a matrix from the coefficients:

$|x\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\begin{pmatrix} x_{00} \\ x_{01} \\ x_{10} \\ x_{11} \end{pmatrix} \longmapsto \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}$$

Then

$$|00\rangle + |11\rangle \mapsto |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

while

$$|v\rangle|w\rangle \longmapsto |v\rangle\langle w|,$$

it is rank-1: maps both $|0\rangle$ and $|1\rangle$ to $|v\rangle$.

The rank of the mx obtained this way is called the <u>Schmidt rank</u> of a vector $|x\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Formally,

<u>Def</u> (Schmidt rank): Let $|x\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.
The Schmidt rank is the smallest $r \in \mathbb{N}$ s.t. we can find $\{|v_i\rangle\}_{i=1}^r \subset \mathcal{H}_A$, $\{|w_i\rangle\}_{i=1}^r \subset \mathcal{H}_B$ s.t.

$$|x\rangle = \sum_{i=1}^{r} |v_i\rangle \otimes |w_i\rangle.$$

Such a decomp. is a min. rank decomposition.

Tensor product of multiple vector spaces:

$\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k$ are Hilbert spaces w/

basis $\{|i\rangle\}_{i=0}^{d_1-1}, \dots \{|i\rangle\}_{i=0}^{d_k-1}$

Then

$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_k$ is a $d_1 \cdots d_k$-dim. space

w/ ONB

$$\left\{ |i_1\rangle \otimes \dots \otimes |i_k\rangle \;\middle|\; i_1 = 0, \dots, d_1 - 1, \dots, i_k = 0, \dots d_{k-1} \right\}$$

For the basis vectors I will also write

$$|i_1\rangle \otimes \dots \otimes |i_k\rangle = |i_1 \dots i_k\rangle$$

Elements of the space:

$$\sum_{i_1 \dots i_k} v_{i_1 \dots i_k} |i_1 \dots i_k\rangle$$

Scalar product:

$$\langle v|w\rangle = \sum_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}} \overline{v}_{i_1 \dots i_k} w_{j_1 \dots j_k} \langle i_1 \dots i_k | j_1 \dots j_k\rangle$$

$$= \sum_{i_1 \dots i_k} \overline{v}_{i_1 \dots i_k} w_{i_1 \dots i_k}.$$

Notice:

(1) $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \cong (\mathcal{H}_A \otimes \mathcal{H}_B) \otimes \mathcal{H}_C \cong \mathcal{H}_A \otimes (\mathcal{H}_B \otimes \mathcal{H}_C)$,

so it is enough to remember the bipartite construction.

(2) $\mathcal{H}_A \otimes \mathcal{H}_B \cong \mathcal{H}_B \otimes \mathcal{H}_A$ via $|ij\rangle \mapsto |ji\rangle$.

Example: $(\mathbb{C}^2)^{\otimes k} = \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{k}$.

Basis: k-long bitstrings:

$$\{|x_1 x_2 .. x_k\rangle\}_{x_j = 0,1}.$$

Size of this basis: 2^k. For $k=3$, this is 8 vectors: $\{|000\rangle, |001\rangle, .., |111\rangle\}$.

Any vector in $(\mathbb{C}^2)^{\otimes 3}$ is written as

$$|v\rangle = v_{000}|000\rangle + v_{001}|001\rangle + .. + v_{111}|111\rangle.$$

$$= \begin{pmatrix} v_{000} \\ v_{001} \\ v_{010} \\ \vdots \\ v_{111} \end{pmatrix} \begin{array}{l} \leftarrow |000\rangle \\ \leftarrow |001\rangle \\ \leftarrow |010\rangle \\ \vdots \\ \leftarrow |111\rangle. \end{array}$$

Tensor prod. of lin. operators:

Def (Tensor product of matrices): Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces, $A \in B(\mathcal{H}_A)$, $B \in B(\mathcal{H}_B)$ (i.e. A, B are matrices). Then their tensor product $A \otimes B \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is the matrix defined by

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle,$$

for all $|v\rangle \in \mathcal{H}_A$, $|w\rangle \in \mathcal{H}_B$.

How does such a mx look like?

$$(A \otimes B)|kl\rangle = \sum_i A_{ik}|i\rangle \otimes \sum_j B_{jl}|j\rangle =$$

$$= \sum_{ij} A_{ik} B_{jl} |ij\rangle, \text{ and thus}$$

$$\langle ij | A \otimes B | kl\rangle = A_{ij} B_{kl}.$$

That is,

$$A \otimes B = \sum_{ijkl} A_{ij} B_{kl} \, |ik\rangle\langle jl| .$$

In matrix notation,

$$
A \otimes B = \begin{array}{cc}
\begin{pmatrix}
A_{\boxed{0}0} B_{\boxed{0}0} & A_{\boxed{0}0} B_{\boxed{0}1} & \cdots & A_{0\,d-1} B_{0\,d-1} \\
A_{\boxed{0}0} B_{\boxed{1}0} & A_{\boxed{0}0} B_{\boxed{1}1} & \cdots & \\
\vdots & \vdots & & \\
A_{d-1,0} B_{d-1,0} & A_{d-1,0} B_{d-1,1} & \cdots & A_{d-1,d-1} B_{d-1,d-1}
\end{pmatrix}
&
\begin{array}{l}
\leftarrow |00\rangle \\
\leftarrow |01\rangle \\
\vdots \\
\leftarrow |d-1,d-1\rangle
\end{array}
\\
\begin{array}{cccc}
\uparrow & \uparrow & & \uparrow \\
|00\rangle & |01\rangle & \cdots & |d-1,d-1\rangle
\end{array}
&
\end{array}
$$

$$
= \begin{pmatrix}
A_{00} \cdot \underline{\underline{B}} & A_{01} \cdot \underline{\underline{B}} & \cdots & A_{0\,d-1} \cdot \underline{\underline{B}} \\
A_{10} \underline{\underline{B}} & \cdots & & \vdots \\
\vdots & & & \\
A_{d-1,0} \underline{\underline{B}} & \cdots & & A_{d-1,d-1} \cdot \underline{\underline{B}}
\end{pmatrix}
$$

Ex.: $X \otimes Z = \begin{pmatrix} X_{00} \cdot Z & X_{01} \cdot Z \\ X_{10} \cdot Z & X_{11} \cdot Z \end{pmatrix} =$

$$= \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

* Eig. vectors / values of elementary tensorprod.:

$(M \otimes N)(|v\rangle \otimes |w\rangle) = \mu \lambda \, |v\rangle \otimes |w\rangle$

if $M|v\rangle = \mu|v\rangle$ and $N|w\rangle = \lambda|w\rangle$

* General op. is of the form $\sum_i M_i \otimes N_i$, in this case no easy way to understand them,

* Tensor product of two self-adj. / unitary mx is self-adj. / unitary !

$(A \otimes B)^+ = A^+ \otimes B^+$, as

$\langle \hat{v} \otimes \hat{w} | (A \otimes B)(v \otimes w) \rangle = \langle \hat{v} \otimes \hat{w} | Av \otimes Bw \rangle$

$= \langle \hat{v} | Av \rangle \langle \hat{w} | Bw \rangle = \langle A^+ \hat{v} | v \rangle \langle B^+ \hat{w} | w \rangle$

$= \langle A^+ \hat{v} \otimes B^+ \hat{w} | v \otimes w \rangle = \langle (A^+ \otimes B^+)(\hat{v} \otimes \hat{w}) | v \otimes w \rangle$

+ works for lin. combination as well.