Quantum framework

* General language used in the description of quantum systems.
* Helps you to understand the language, but does not tell how to actually model realistic systems.

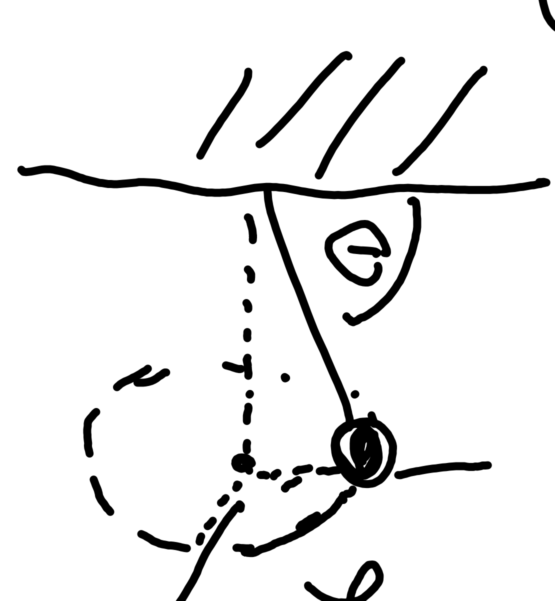
Components for framework

I.e., What do we need to describe?
2 examples: classical mechanics and probability theory.

Classical mechanics

: Want to understand: under the application of forces, how do things move.

#1 Degrees of freedom.



Pendulum fixed to ceiling Can be described by 2 angles, θ and φ.

#2: State: to describe the system, you need (φ, Θ).

#3: Time evolution: Want to understand how $(\varphi(t), \Theta(t))$ looks like.

Method: forces $\Rightarrow$ 2^{nd} order diff eq $\Rightarrow$ solve $\rightarrow$ $(\varphi(t), \Theta(t))$.

#4: Solve actual questions: Eg. how high is the pendulum @ time t

$$h(t) = \left[1 - \sin(\Theta(t))\right] \cdot \text{length}.$$

(This is a bit forced, but the pt is that you need to do sth w/ the state to get info out)

Quantum mechanics

What we learn: language for describing discrete quantum systems.

Discrete: energy levels of atoms, polarization of photon, etc.

Components:

0) Degrees of freedom - what configurations can the system be in

1) State - describing the config. at a given time

2) Time evolution - how the state changes in time

3) Measurement - gaining classical info from quantum system.
Meas. changes the state, so describing the post-measurement state is also important.

The rules

0) DoF – to every quantum system we assign a Hilbert space H. If the system can be in d different configuration, then $\mathcal{H} = \mathbb{C}^d$.
To composite systems we assign the tensor product of the two Hilbert spaces describing the two components.

1) State of the system: equivalence classes of normalized vectors:
$|\psi\rangle \in \mathcal{H}$, $\|\psi\| = 1$, $|\psi\rangle \equiv e^{i\varphi}|\psi\rangle$, $\varphi \in \mathbb{R}$.

2) Time evolution: described by a unitary operator.
$|\psi(t)\rangle = \mathcal{U}(t) \cdot |\psi(0)\rangle$.

3) Measurements: have to describe

- possible outcomes (classical info)
- measurement is inherently probabilistic, so the probability w/ which each outcome occurs.
- The state after the measurement. This depends on the outcome.

A measurement w/ n outcomes $1, 2, \dots, n$ is given by operators $\{M_i\}_{i=1}^n$ s.t.

$$\sum_i M_i^\dagger M_i = 1.$$

The outcome i happens w/ probability

$$p_i = \| M_i |\psi\rangle \|^2$$

The state after the measurement is

$$\frac{M_i |\psi\rangle}{\| M_i |\psi\rangle \|}$$

This general form of meas. is called a POVM = positive operator valued measure (as $M_i^\dagger M_i$ defines such a measure).

Examples:

Degrees of freedom ($\mathcal{H}$ Hilbert space):

- photon's polarization : H/V $\rightarrow$ 2 DOF, $\mathcal{H} = \mathbb{C}^2$.
- atom: ground/excited state $\rightarrow$ 2 DOF, $\mathcal{H} = \mathbb{C}^2$.

Sometimes we name the computational basis according to their meaning, e.g., $|H\rangle$ and $|V\rangle$ for photon polarization or $|\uparrow\rangle$ and $|\downarrow\rangle$ for spins (i.e., magnets attached to) electrons.

If DOF = 2 : we call such a system a QUBIT (quantum bit)

Composite systems are tensor products:

Given 2 photons, each can be H/V, thus the two photons together can be in the state HH, HV, VH, VV $\rightarrow$ this is exactly the basis of the tensor prod. space.

States: normalized vectors in $\mathcal{H}$, up to a phase. If $\mathcal{H} = \mathbb{C}^2$, then

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \text{with}$$

$$1 = \langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2.$$

Let $\theta \in [0, \pi]$ be such that

$$|\alpha| = \cos\frac{\theta}{2}, \quad |\beta| = \sin\left(\frac{\theta}{2}\right).$$

Then we can write $\alpha = e^{i\chi}\cos\frac{\theta}{2}$

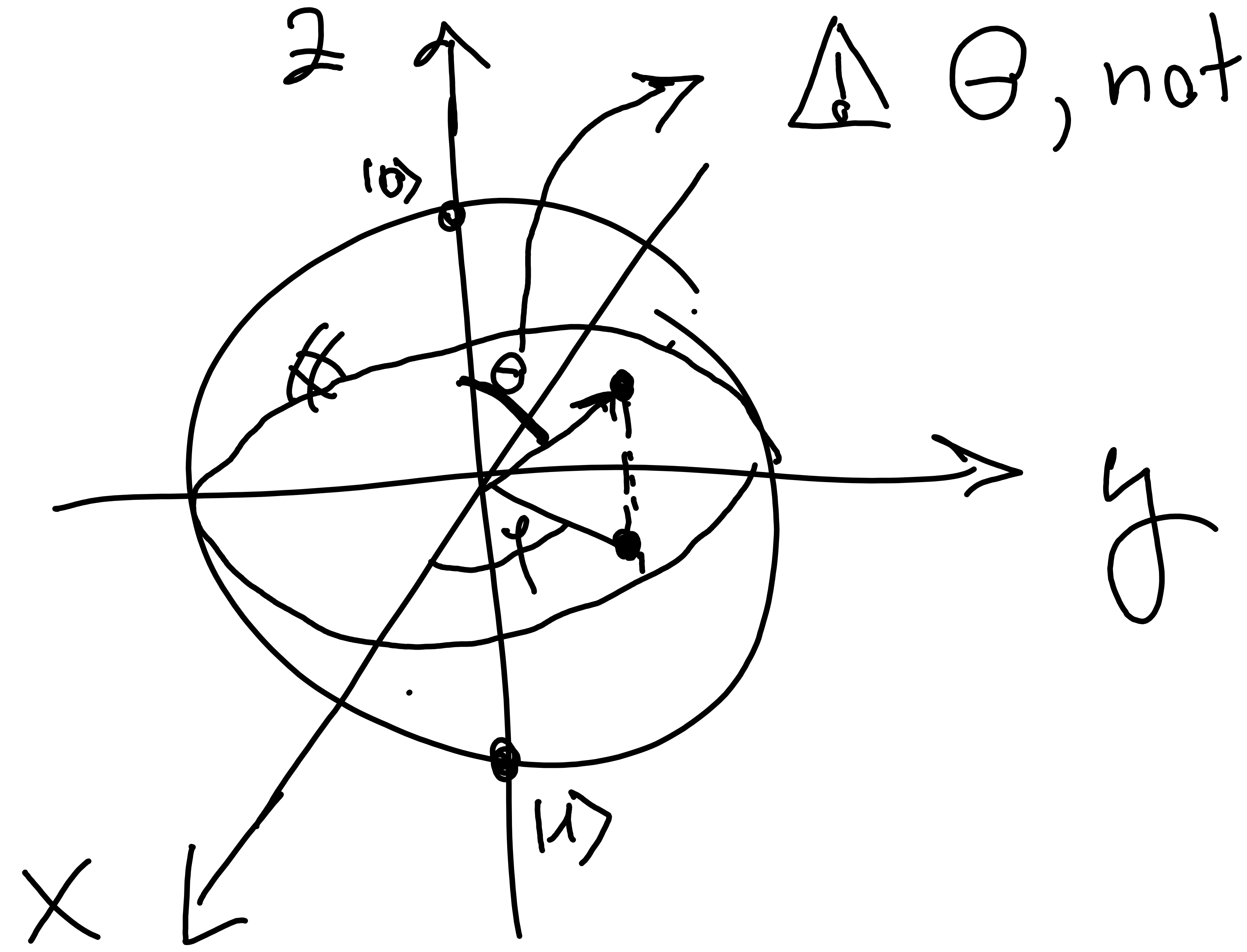
$$\beta = e^{i(\chi+\varphi)}\sin\left(\frac{\theta}{2}\right),$$

and thus

$$|\psi\rangle = e^{i\chi}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right).$$

Here $\varphi \in [0, 2\pi)$ and $\theta \in [0, \pi]$, and the phase χ is irrelevant (vectors w/ same φ, θ but different χ represent the same quantum state).

<u>We can depict the states on a sphere:</u>



⚠ θ, not $\theta/2$.

There is a 1-to-1 correspondence btw pts on the surface of the sphere and quantum states in 2D $\Rightarrow$ Visualization of QUBITS.

<u>Special points:</u>

- $z = (0,0,1) \Leftrightarrow \theta = 0 \Rightarrow |0\rangle$
- $-z = (0,0,-1) \Leftrightarrow \theta = \pi \Rightarrow |1\rangle$

} eig. vectors of Z

- $x = (1,0,0) \Leftrightarrow \theta = \frac{\pi}{2}, \varphi = 0 \Rightarrow |+\rangle$
- $-x = (-1,0,0) \Leftrightarrow \theta = \frac{\pi}{2}, \varphi = \pi \Rightarrow |-\rangle$

} eig. vectors of X

- $y = (0,1,0) \leftrightarrow \theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2} \Rightarrow |0\rangle + i|1\rangle$
- $-y = (0,-1,0) \leftrightarrow \theta = \frac{\pi}{2}, \varphi = -\frac{\pi}{2} \Rightarrow |0\rangle - i|1\rangle$

$\Big\}$ eig. vectors of Y.

<u>Remark</u> Convenient way to get rid of the phase is considering $\rho = |\psi\rangle\langle\psi|$.

This is a $d \times d$ mx, $tr(\rho) = \langle\psi|\psi\rangle = 1$. ⚠

If $|\varphi\rangle = e^{i\chi}|\psi\rangle$, $\chi \in \mathbb{R}$, then $\langle\varphi| = e^{-i\chi}\langle\psi|$, and thus $|\varphi\rangle\langle\varphi| = |\psi\rangle\langle\psi|$.

HW Calculate $|\psi\rangle\langle\psi|$ w/ the previous parametrization. Use trigonometrical identities until only θ appears in the formulas, not $\frac{\theta}{2}$. Exand the result in the Pauli basis. What do you observe?

States in composite system:

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can either have the form $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, or it is not of this form (instead, a linear combination of such vectors).

If $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$: we call this a product state.

If $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$: we call such a state ENTANGLED.

Time evolution: w/ unitaries.

We consider discrete time steps: Unitaries ating one after the other. We also call them gates.

Example gates: X, Y, Z.

$$X: |0\rangle \mapsto |1\rangle$$
$$|1\rangle \mapsto |0\rangle$$

X eigenstates: $|+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$
normalized
$|-\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

change as $X|+\rangle = |+\rangle$

$X|-\rangle = -|-\rangle \equiv |-\rangle$

We can only do $-|\psi\rangle \equiv |\psi\rangle$ when $|\psi\rangle$ describes a state. In calculations usually we write lin. comb. of vectors, there keep the sign!

Note: $XZ = iY$, so Y can be obtained from X & Z only.

One more important gate: Hadamard gate,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

H is self-adjoint, so $H^2 = HH^\dagger = \mathbb{1}$.

$H|0\rangle = |+\rangle$, $H|+\rangle = |0\rangle$

$H|1\rangle = |-\rangle$, $H|-\rangle = |1\rangle$.

It changes between eigenstates of X and Z. In other words,

$X = HZH$.

Composite systems: tensor prod. of unitaries is unitary. E.g. $H^{\otimes n}$ is unitary. It acts as: $H^{\otimes n}|0\rangle^{\otimes n} = |+\rangle^{\otimes n} = \sum_{\underline{x} \in \{0,1\}^n} |\underline{x}\rangle$.

def. of tensor prod. of matrices

Not every unitary is tensor product. E.g.

$$\text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$$

$$= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$$

is not a tensor product. Such gates can bring product states to entangled states, unlike product gates.

In classical computation, copying is an essential operation.

This is not possible in Q.M :

<u>Thm</u> (No-cloning theorem) : There is no unitary $\mathcal{H}\otimes\mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ s.t.

$$|0\rangle|\psi\rangle \mapsto |\psi\rangle|\psi\rangle \quad \forall |\psi\rangle \in \mathcal{H}.$$

<u>Remark</u> $|0\rangle$ is needed as we need unitary, can't grow the dimension!

<u>Proof</u> : By contradiction. Assume such a unitary (actually, lin. map) exists.

$$\mathcal{U} : |00\rangle \mapsto |00\rangle$$

$$\mathcal{U} : |01\rangle \mapsto |11\rangle$$

Then by linearity,

$$\mathcal{U} : |0+\rangle \mapsto \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

And by assumption

$$\mathcal{U} : |0+\rangle \mapsto |++\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

$$\neq \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

This is a contradiction, and thus there is no such lin. map $\square$

—o—

Recap: rules of QM.

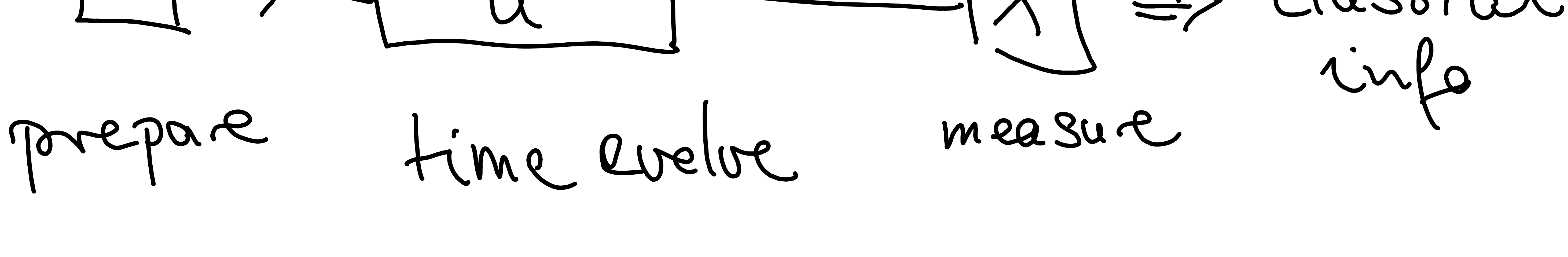
- To $\forall$ quant. system we assign a $\mathcal{H}$-space. To a system composed of 2 smaller ones we assign the tensor product.

- States are eq. classes of normalized vectors, eg.: $|\psi\rangle \cong \lambda|\psi\rangle$ if $|\lambda| = 1$.

- Time evolution: unitaries

- Measurement: $\{M_i\}_{i=1}^n$, $\sum_i M_i^\dagger M_i = \mathbb{1}$,
  - $p_i = \| M_i |\psi\rangle \|^2$
  - $\psi_i = \dfrac{M_i|\psi\rangle}{\| M_i |\psi\rangle \|}$.

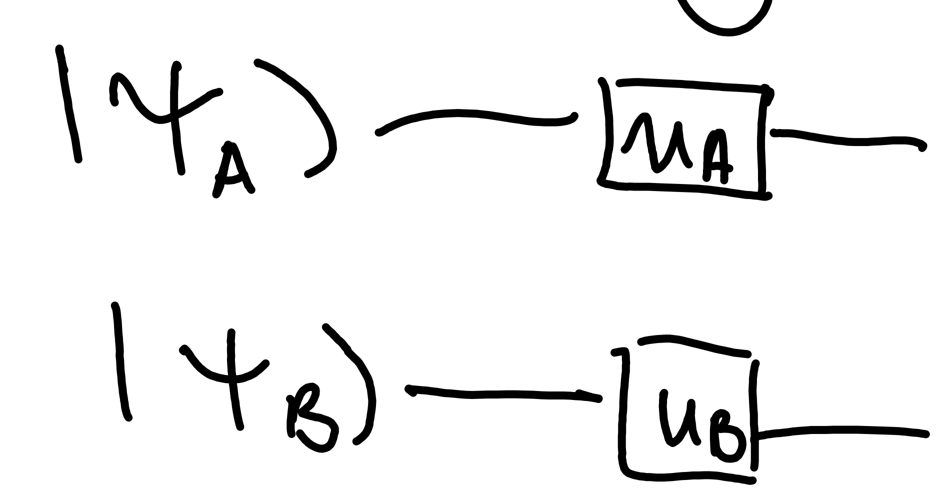
Convenient picture:



Time evolution:

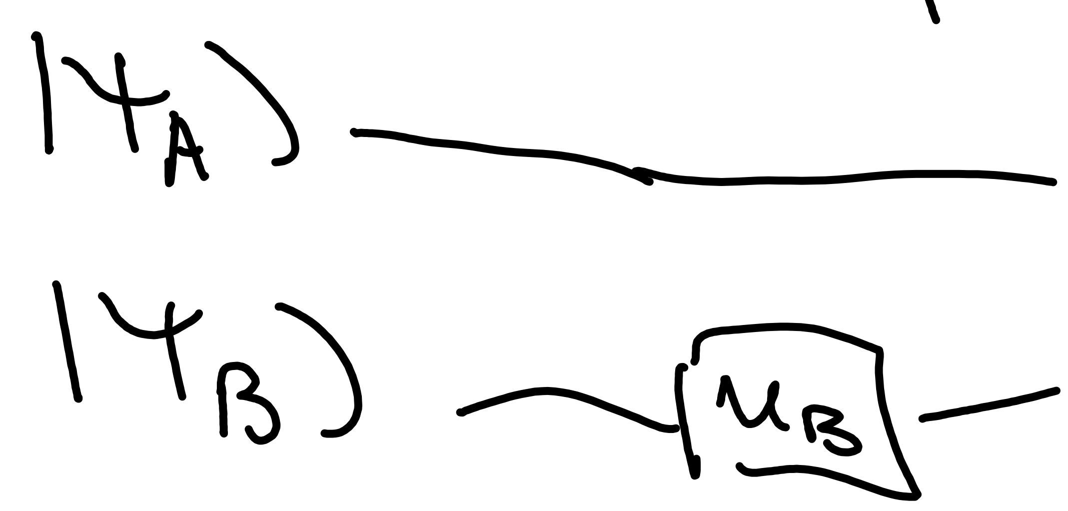
(1) No-cloning thm: $U: |\psi\rangle|0\rangle \mapsto |\psi\rangle|\psi\rangle \;\forall |\psi\rangle$ not possible. Remark: classical cloning $|00\rangle \mapsto |00\rangle, |10\rangle \mapsto |11\rangle$ OK.
HW: Create such a unitary!

(2) On a composite system $\mathcal{H}_A \otimes \mathcal{H}_B$, $U_A \otimes U_B$ is unitary if U_A, U_B are. A graph. notation:



$$(U_A \otimes U_B)(|\psi_A\rangle \otimes |\psi_B\rangle)$$

We say "A does nothing, B evolves" for $U_A = \mathbb{1}$. Graphically,



$$(\mathbb{1} \otimes U_B)(|\psi_A\rangle \otimes |\psi_B\rangle).$$

(3) Not every unitary is of this form, e.g.

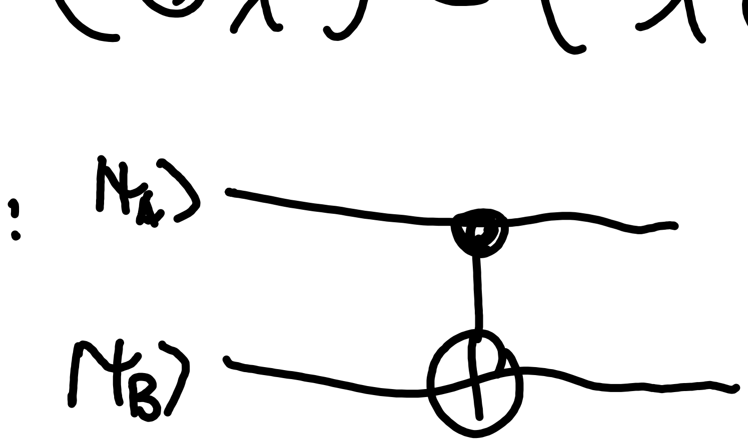
$CNOT = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$.

→ A sees 0 ⇒ B does nothing

→ A sees 1 ⇒ B negates

$$CNOT = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

A graph. notation:



Measurements

- outcomes : $1, 2, \dots, n$.
- meas. operators : $M_1, M_2, \dots, M_n \in \mathcal{B}(\mathcal{H})$ s.t.

  $$\sum_i M_i^\dagger M_i = \mathbb{1}.$$

- Then i happens w/ proba

  $$p_i = \| M_i |\psi\rangle \|^2$$

- And post-meas. state is

  $$|\psi_i\rangle = \frac{M_i|\psi\rangle}{\| M_i |\psi\rangle \|}.$$

Remark: $\sum_i M_i^\dagger M_i \Rightarrow \sum_i p_i = 1$.

Proof: $p_i = \| M_i |\psi\rangle \|^2 = \langle \psi | M_i^\dagger M_i |\psi\rangle$

$$\sum_i p_i = \sum_i \langle \psi | M_i^\dagger M_i |\psi\rangle = \langle \psi | \sum_i M_i^\dagger M_i |\psi\rangle =$$

$$= \langle \psi | \mathbb{1} |\psi\rangle = \langle \psi | \psi\rangle = 1. \quad \square$$

So you can remember the $\sum_i M_i^\dagger M_i = \mathbb{1}$ condition.

Examples $\mathcal{H} = \mathbb{C}^2$. Let us consider

$$M_0 = |0\rangle\langle 0|, \quad M_1 = |1\rangle\langle 1|.$$

Then $M_0^\dagger M_0 = M_0^2 = M_0$ and $M_1^\dagger M_1 = M_1^2 = M_1$ and

$$M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{1}.$$

Let's do this measurement on $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Then the outcome

- 0 appears w/ proba $\| M_0 |\psi\rangle \|^2 = |\alpha|^2$.

  the post-meas. state is $\frac{M_0|\psi\rangle}{\| M_0|\psi\rangle\|} = |0\rangle$.

- 1 appears w/ proba $\| M_1 |\psi\rangle \|^2 = |\beta|$.

  the post-meas. state is $\frac{M_1|\psi\rangle}{\| M_1|\psi\rangle\|} = |1\rangle$.

If M_i are projecters, we call this a <u>projective</u> measurement.

A convenient way to package measurement in compact form:

Take a Hermitian operater w/ eigen projecters $M_0, M_1, \ldots, M_n$, that is,

$$O = \sum_i \lambda_i M_i .$$

Then we say "measure O" instead of saying measure the PeVM $M_0, M_1, \ldots, M_n$.

Eg. for $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

$Z = M_0 - M_1$ → measuring the comp. basis can bee also called "measure Z".

⚠ $Id = M_0 + M_1$, but this is not the eigen decomposition, so it does not specify this measurement. ⚠

Eg. #2 $M_0 = |+\rangle\langle +|$, $M_1 = |-\rangle\langle -|$: here,

$X = |+\rangle\langle +| - |-\rangle\langle -|$,

so we call this measurement "measuring X".

Self-adjoint operaters are alse called observables.

Remark: Expectation value:

$$\langle O \rangle_\psi := \langle \psi | O | \psi \rangle = \sum_i \lambda_i \langle \psi | P_i | \psi_i \rangle$$
$$= \sum_i \lambda_i p_i,$$

so this is the expectation value of the random variable specified by the eig. values of O.

Remark: Consider $|\psi\rangle = |0\rangle$. Measuring Z leads to result "+1" w/ probability 100%, as it is an eigenstate.

Measuring X on $|0\rangle$ leads to +1 w/ probability 50% and to −1 w/ proba 50%.

As X and Z does not commute, they do not have a joint eigenbasis $\Rightarrow$ "can't measure them together".

Measurement in composite systems:

→ It is still just a set of operators

$M_i \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ s.t. $\sum_i M_i^\dagger M_i = \mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B$.

→ Special measurements:

"A measures $\{M_i\}_{i=1}^m$ B measures $\{N_i\}_{i=1}^n$":

This refers to a measurement of the form

$$\{X_{ij} = M_i \otimes N_j \mid i = 1..m,\ j = 1..n\}$$

Remember that

$$\sum_{ij} (M_i^\dagger \otimes N_j^\dagger)(M_i \otimes N_j) =$$

$$= \sum_{ij} M_i^\dagger M_i \otimes N_j^\dagger N_j = \sum_i M_i^\dagger M_i \otimes \sum_j N_j^\dagger N_j = \mathbb{1},$$

So these operators do specify a meas.

→ In particular, A measures $\{M_i\}_{i=1}^m$

denotes the meas. $\{M_i \otimes \mathbb{1}\}_{i=1}^m$.

Similarly, B meas. M_i denotes the meas. $\{\mathbb{1} \otimes M_i\}_{i=1}^m$.

Eg: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and A measures Z.

This encodes the measurement

$$M_0 = |0\rangle\langle 0| \otimes \mathbb{1}, \quad M_1 = |1\rangle\langle 1| \otimes \mathbb{1}.$$

Outcome "+1", labelled "0", happens w/

$$p_0 = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = \frac{1}{2}.$$

The post-meas. state is

$$\psi_0 = \frac{M_0|\psi\rangle}{\| M_0|\psi\rangle \|} = |00\rangle.$$

Outcome "−1", labelled "1", happens w/

$$p_1 = \langle\psi|M_1^\dagger M_1|\psi\rangle = \langle\psi|M_1|\psi\rangle = \frac{1}{2}.$$

The post-meas. state is

$$\psi_1 = |11\rangle.$$

$\boxed{\text{HW}}$ Take the 2 post-meas. state and let B measure Z in both cases. What do you find?

In another scenario, let A and B both measure Z. That is, they perform the measurement

$$M_{00} = |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \quad M_{01} = |0\rangle\langle 0| \otimes |1\rangle\langle 1|$$

$$M_{10} = |1\rangle\langle 1| \otimes |0\rangle\langle 0|, \quad M_{11} = |1\rangle\langle 1| \otimes |1\rangle\langle 1|.$$

Calculate the outcome probabilities and post-meas. states! Compare it w/ the previous setup!

<u>Note</u> you should find: measuring the same time is exactly the same as measuring one after the other.