Density mx formalism

Let $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Question: is there a state on $\mathcal{H}_A$ s.t. measuring $|\psi_{AB}\rangle$ on A is the same as measuring $|\psi_A\rangle$?

Measuring $|\psi_{AB}\rangle$ on A: given by $\{M_i\}_{i=1}^{n} \subseteq B(\mathcal{H}_A)$ s.t. $\sum_i M_i^\dagger M_i = \mathbb{1}$. Measurement outcomes:

$$p_i = \langle \psi_{AB} | M_i^\dagger M_i \otimes \mathbb{1} | \psi_{AB}\rangle ,$$

We want thus

$$\langle \psi_{AB} | M_i^\dagger M_i \otimes \mathbb{1} | \psi_{AB}\rangle = \langle \psi_A | M_i^\dagger M_i | \psi_A\rangle \ \forall i$$

$\forall$ measurement. Let O be any Hermitian operator, $O = \sum_i \lambda_i P_i$, w/ $P_i^\dagger P_i = P_i^2 = P_i$, $\sum_i P_i = \mathbb{1}$: this is a measurement. Then, by assumption,

$$\langle \psi_{AB} | P_i \otimes \mathbb{1} | \psi_{AB}\rangle = \langle \psi_A | P_i | \psi_A\rangle ,$$

and thus the exp. values coincide:

$$\langle \psi_{AB} | O \otimes \mathbb{1} | \psi_{AB}\rangle = \sum_i \lambda_i \langle \psi_{AB} | P_i \otimes \mathbb{1} | \psi_{AB}\rangle =$$

$$= \sum_i \lambda_i \langle \psi_A | P_i | \psi_i \rangle = \langle \psi_A | O | \psi_A \rangle .$$

We thus are looking for $|\psi_A\rangle$ s.t.

$$\langle \psi_{AB} | O \otimes \mathbb{1} | \psi_{AB} \rangle = \langle \psi_A | O | \psi_A \rangle .$$

<u>Remark:</u> $M_i^+ M_i$ is also self-adjoint and thus $\langle O_A \rangle = \langle \psi_{AB} | O \otimes \mathbb{1} | \psi_{AB} \rangle$ contains as a special case

$$p_i = \langle \psi_{AB} | M_i^+ M_i \otimes \mathbb{1} | \psi_{AB} \rangle .$$

<u>Prop:</u> Let $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$. Then

$$\langle \psi_{AB} | O \otimes \mathbb{1} | \psi_{AB} \rangle = \frac{1}{2} \mathrm{tr}\{O\}$$

<u>Proof:</u> direct calculation:

$$\langle \psi | O \otimes \mathbb{1} | \psi \rangle = \frac{1}{2} \left(\langle 00| + \langle 11| \right) O \otimes \mathbb{1} \left(|00\rangle + |11\rangle \right)$$

$$= \frac{1}{2} \left(\langle 0 | O | 0 \rangle + \langle 1 | O | 1 \rangle \right) = \frac{1}{2} \mathrm{tr}\{O\} .$$

$\square$

This can be repeated for all states.

Prop: Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then there is a unique matrix $\rho_A \in B(\mathcal{H}_A)$ s.t.

$$\mathrm{tr}\{\rho_A O\} = \langle\psi| O \otimes \mathbb{1} |\psi\rangle \quad \forall O \in B(\mathcal{H}_B).$$

Proof: Existence:

$\langle\psi| O \otimes \mathbb{1} |\psi\rangle = \mathrm{tr}\{\rho (O \otimes \mathbb{1})\}$, where $\rho = |\psi\rangle\langle\psi| \in B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$. Let us write $\rho = \sum_i A_i \otimes B_i$, e.g. min. rank decomp. Then

$$\mathrm{tr}\{\rho\, O \otimes \mathbb{1}\} = \sum_i \mathrm{tr}\{A_i O \otimes B_i\} =$$

$$= \sum_i \mathrm{tr}\{A_i O\} \cdot \mathrm{tr}\{B_i\} = \mathrm{tr}\{\rho_A \cdot O\},$$

where

$$\rho_A = \sum_i A_i \cdot \mathrm{tr}\{B_i\}.$$

Uniqueness: clear from ex. class: $\mathrm{tr}(A^\dagger B)$ is scalar prod. on Hermitian ops, so

$$\mathrm{tr}\{\rho_A^\dagger O\} = \mathrm{tr}\{\tilde{\rho}_A^\dagger O\} \quad \forall O \in B(\mathcal{H}),\ O = O^\dagger \Rightarrow$$

$$\rho_A = \tilde{\rho}_A. \qquad \square$$

Definition (partial trace): Let $M \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$, $M = \sum_i N_i \otimes K_i$ be a matrix. Then its partial traces $\mathrm{tr}_A(M) \in \mathcal{B}(\mathcal{H}_B)$ and $\mathrm{tr}_B(M) \in \mathcal{B}(\mathcal{H}_A)$ are

$$\mathrm{tr}_A(M) = \sum_i \mathrm{tr}\{N_i\} K_i \text{, and}$$

$$\mathrm{tr}_B(M) = \sum_i N_i \cdot \mathrm{tr}\{K_i\}.$$

Remark:

$$\sum_i A^i \otimes B^i = \begin{pmatrix} A^i_{00} B^i & \cdots & A^i_{0,d-1} B^i \\ \vdots & & \\ A^i_{d-1,0} B^i & \cdots & A^i_{d-1,d-1} B^i \end{pmatrix} \text{, so}$$

to calculate tr_A: sum the blocks in the diagonal.

to calculate tr_B: trace each of the blocks.

Example:

$$X = \left(\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right) \in \mathcal{M}_2 \otimes \mathcal{M}_2 = \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2)$$

$$\mathrm{tr}_A X = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 11 & 12 \\ 15 & 16 \end{pmatrix} = \begin{pmatrix} 12 & 14 \\ 20 & 22 \end{pmatrix}$$

$$\mathrm{tr}_B X = \left(\begin{array}{c|c} 1+6 & 3+8 \\ \hline 9+14 & 11+16 \end{array}\right) = \begin{pmatrix} 7 & 11 \\ 23 & 27 \end{pmatrix}$$

Remark: $\mathrm{tr} \circ \mathrm{tr}_A = \mathrm{tr} \circ \mathrm{tr}_B = \mathrm{tr}$

Example: If $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then $\mathrm{tr}_A |\psi\rangle\langle\psi| = \mathrm{tr}_B |\psi\rangle\langle\psi| = \frac{1}{2}\mathbb{1}$.

if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then $\mathrm{tr}_B |\psi\rangle\langle\psi| = |\psi_A\rangle\langle\psi_A| \cdot \mathrm{tr}\{|\psi_B\rangle\langle\psi_B|\}$
$= |\psi_A\rangle\langle\psi_A|$, and $\mathrm{tr}_A |\psi\rangle\langle\psi| = |\psi_B\rangle\langle\psi_B|$.

So we have found matrices that carry all information (i.e., all meas. outcome probabilities) about "half of a state". What are the properties of such a matrix?

Prop: Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\||\psi\rangle\| = 1$. Let

$$\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|). \text{ Then}$$

$$\mathrm{tr}(\rho_A) = 1 \text{ and } \rho_A \geq 0.$$

Proof: If $|\psi\rangle\langle\psi| = \sum_i A_i \otimes B_i$, then $\rho_A = \sum_i A_i \cdot \mathrm{tr}(B_i)$

and $\mathrm{tr}\{\rho_A\} = \sum_i \mathrm{tr}\{A_i\}\mathrm{tr}\{B_i\} = \sum_i \mathrm{tr}\{A_i \otimes B_i\}$

$= \mathrm{tr}\{|\psi\rangle\langle\psi|\} = \langle\psi|\psi\rangle = 1.$

To see positivity, note that if

$|\psi\rangle\langle\psi| = \sum_i A_i \otimes B_i$, then

$$\rho_A = \sum_j \left(\sum_i A_i \cdot \langle j|B_i|j\rangle \right).$$

Then

$$\langle x|\rho_A|x\rangle = \sum_j \sum_i \langle x|A_i|x\rangle \cdot \langle j|B_i|j\rangle$$

$$= \sum_j \left| \left(\langle x| \otimes \langle j| \right) |\psi\rangle \right|^2 \geq 0,$$

so $\rho_A \geq 0$. $\square$

<u>Definition</u> (<u>Density matrix</u>): A matrix $\rho \in B(\mathcal{H})$ s.t. $\mathrm{tr}(\rho) = 1$ and $\rho \geq 0$, i.e., ρ positive semi-definite, is called a <u>density matrix</u>.

<u>Example</u>: given $|\psi\rangle \in \mathcal{H}$, $|\psi\rangle\langle\psi| \in B(\mathcal{H})$ is a density matrix:

$$\mathrm{tr}\{ |\psi\rangle\langle\psi| \} = \sum_i \langle i|\psi\rangle\langle\psi|i\rangle =$$

$= \sum_i \langle\psi|i\rangle\langle i|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = \langle\psi|\psi\rangle = 1.$

(Note: tr is cyclic: $\mathrm{tr}\{AB\} = \mathrm{tr}\{BA\}$ for any two matrix – even if they are $1\times n$ and $n\times 1$).

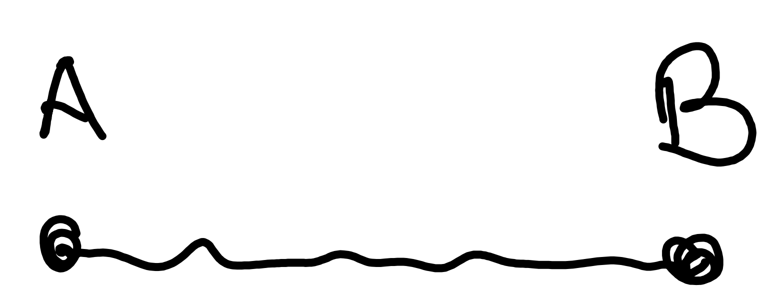
Also, $|\psi\rangle\langle\psi| \geq 0$ as $\langle\phi|\psi\rangle\langle\psi|\phi\rangle = |\langle\phi|\psi\rangle|^2 \geq 0$ $\forall |\phi\rangle \in \mathcal{H}$.

We will transition from vector description to density matrices $\rightsquigarrow$ state = density matrix. These states will be called <u>pure</u> states.

<u>Example 2</u>: $\rho = \frac{1}{2}\mathbb{1}$ $\mathrm{tr}\{\rho\} = 1$ and $\rho \geq 0$.

We will call this the maximally mixed state.

<u>Probabilistic interpretation of the density matrix</u>:



$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$

Imagine B measures in the z basis, but does not tell the result to A.

What is the state that A has?

Measure in Z basis: $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Outcome +1 w/ probability

$$\langle \psi | (\mathbb{1} \otimes |0\rangle\langle 0|) | \psi \rangle = \frac{1}{2}$$

Outcome -1 w/ probability

$$\langle \psi | (\mathbb{1} \otimes |1\rangle\langle 1|) | \psi \rangle = \frac{1}{2}.$$

The post-meas. states are

$$|\psi_+\rangle = \frac{1}{\text{norm}} \cdot (\mathbb{1} \otimes |0\rangle\langle 0|) |\psi\rangle = |00\rangle$$

$$|\psi_-\rangle = |11\rangle.$$

So A has the state $|0\rangle$ w/ probability 1/2 and the state $|1\rangle$ w/ probability 1/2.

This is called an <u>ensemble</u>:

$$\left\{ \left(\tfrac{1}{2}, |0\rangle\right), \left(\tfrac{1}{2}, |1\rangle\right) \right\}.$$

Assume that A measures now O. What is

$$\langle O \rangle = ?$$

$$\langle O \rangle = \frac{1}{2} \cdot \langle 0|O|0\rangle + \frac{1}{2} \langle 1|O|1\rangle = \mathrm{tr}\left\{ O \, \tfrac{1}{2} \mathbb{1} \right\}.$$

So what A knows: the density mx

$$\frac{1}{2}\mathbb{1} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| .$$

<u>Note</u>: it is the same density matrix as before! If B measures but does not reveal the meas. outcome $\Rightarrow$ no extra information!

More explicitly: assume now B measures in the X basis. That is: $X = |+\rangle\langle +| - |-\rangle\langle -|$.

Then

$$p_+ = \langle\psi| (\mathbb{1}\otimes|+\rangle\langle +|)|\psi\rangle =$$

$$= \|\underbrace{(\mathbb{1}\otimes\langle +|)}_{\mathbb{C}^4\to\mathbb{C}^2 \text{ matrix}}|\psi\rangle\|^2 = \frac{1}{4}\left\|\left(\mathbb{1}\otimes\langle 0| + \mathbb{1}\otimes\langle 1|\right)\left(|00\rangle + |11\rangle\right)\right\|^2$$

$$= \frac{1}{4}\cdot 2 = \frac{1}{2}.$$

$$p_- = \frac{1}{2}.$$

The post-meas. state is

$$|\psi_+\rangle = \frac{1}{\text{norm}}\cdot(\mathbb{1}\otimes|+\rangle\langle +|)|\psi\rangle) = |+\rangle\otimes|+\rangle$$

$$|\psi_-\rangle = \frac{1}{\text{norm}}(\mathbb{1}\otimes|-\rangle\langle -|)|\psi\rangle = |-\rangle\otimes|-\rangle$$

So A has the state $|+\rangle$ w/ 1/2 proba, $|-\rangle$ w/ 1/2 proba.

That is, the ensemble

$$\left\{ \left(\frac{1}{2}, |+\rangle\right), \left(\frac{1}{2}, |-\rangle\right) \right\}.$$

Let us do a meas.! Wlog again take exp. values of $O \in \mathcal{B}(\mathcal{H}_A)$, $O = O^\dagger$.

$$\langle O \rangle = \frac{1}{2} \cdot \langle + | O | + \rangle + \frac{1}{2} \langle - | O | - \rangle =$$

$$= \operatorname{tr}\left\{ O \left(\frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |-\rangle\langle -| \right) \right\} = \operatorname{tr}\left\{ O \, \frac{1}{2} \mathbb{1} \right\}.$$

So there is no measurement that differentiates between

$$\left\{ \left(\frac{1}{2}, |+\rangle\right), \left(\frac{1}{2}, |-\rangle\right) \right\} \text{ and } \left\{ \left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right) \right\}.$$

The information we can get (through measurements — the only way to get info from a quantum system) is exactly the <u>density matrix</u> corresponding to the ensemble,

$$\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|.$$

We have also seen that different ensembles can give rise to the same density matrix!

Def: Let $\rho \in B(\mathcal{H})$ be a density matrix. Then an ensemble decomposition is given by $n \in \mathbb{N}$, $p \in \mathbb{R}_+^n$, $\{|\psi_i\rangle\}_{i=1}^n$ s.t. $\sum_i p_i = 1$ and $\| |\psi_i\rangle \|^2 = 1$ $\forall i = 1..n$, and

$$\rho = \sum_{i=1}^n p_i |\psi_i\rangle\langle\psi_i| .$$

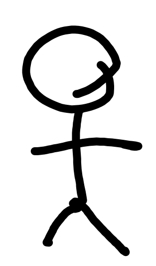
All ensembles $\{(p_i, |\psi_i\rangle)\}$ w/ $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ contain the same information about the system (and it is the density matrix ρ).

$\rho \to$ our knowledge about the system.

If $\rho = |\psi\rangle\langle\psi|$, pure state, then we have full knowledge (note that meas. outcomes are still probabilistic!

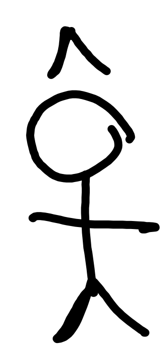
$\rho \neq |\psi\rangle\langle\psi| \approx$ missing knowlegde.

Example: state preparati



50% → $|0\rangle$

50% → $|1\rangle$



50% → $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

50% → $\frac{1}{\sqrt{2}}(\underbrace{|0\rangle - |1\rangle}_{|-\rangle}) = |-$

Can you tell the difference w/ a measurement?

Density matrices

Consider a state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Let

$$\rho_A = \mathrm{tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|) = (\mathrm{id} \otimes \mathrm{tr})(|\psi_{AB}\rangle\langle\psi_{AB}|)$$
$$\rho_B = \mathrm{tr}_A(|\psi_{AB}\rangle\langle\psi_{AB}|) = (\mathrm{tr} \otimes \mathrm{id})(|\psi_{AB}\rangle\langle\psi_{AB}|)$$

Here, $\mathrm{tr}_A : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$, $\mathrm{tr}_A = \mathrm{tr} \otimes \mathrm{id}$ and
$\mathrm{tr}_B : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$, $\mathrm{tr}_B = \mathrm{id} \otimes \mathrm{tr}$.

That is, $\mathrm{tr}_A : \sum_i M_i \otimes N_i \mapsto \sum_i \mathrm{tr}(M_i) \cdot N_i$
$\mathrm{tr}_B : \sum_i M_i \otimes N_i \mapsto \sum_i M_i \, \mathrm{tr}(N_i)$.

By definition: $\langle\psi_{AB}| O \otimes \mathrm{id} |\psi_{AB}\rangle = \mathrm{tr}\{\rho_A O\}$
Contains all info about the state A has access to.
We have seen $\rho_A \geq 0$, $\mathrm{tr}(\rho_A) = 1$.

$$S(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq 0, \ \mathrm{tr}\rho = 1\}$$

These matrices are called density matrices.

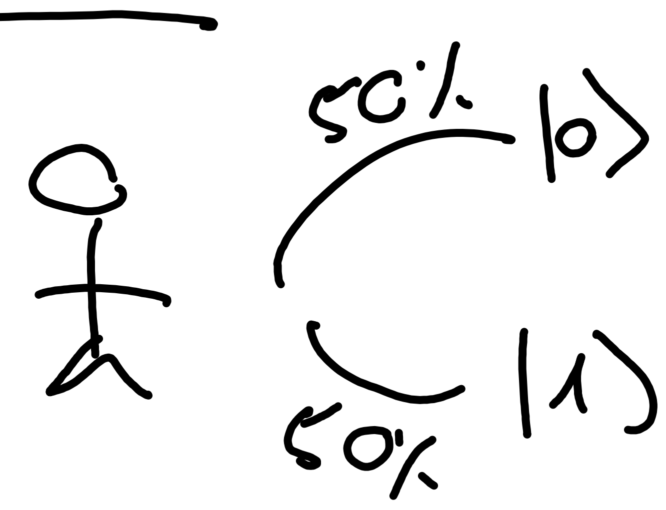
Note: $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \not\geq 0$ as $(1 \ -i)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 2+i \not\geq 0$,
as $2+i \notin \mathbb{R}$.

Ensemble interpretation: if B measures but doesn't tell the outcome, A has exactly the same knowledge $\Rightarrow$ same density matrix! If A knows the

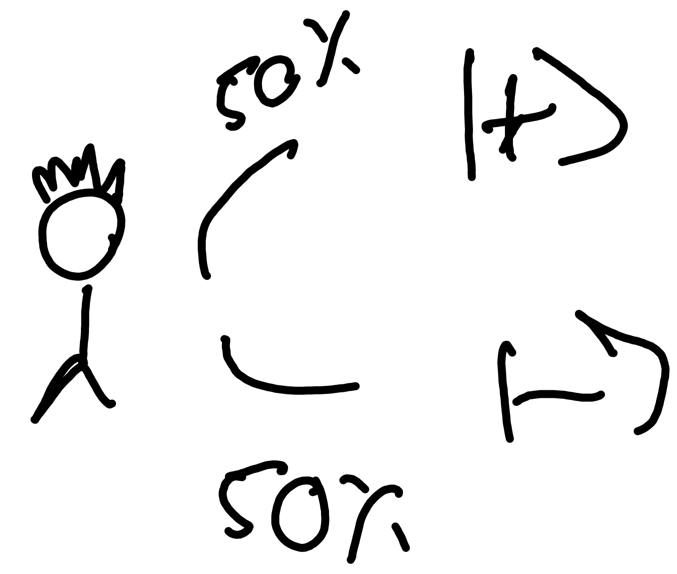
Meas. basis: gives rise to an ensemble interpretation of its density matrix,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| .$$

<u>Ex:</u>



vs.



$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\mathbb{1} = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| ,$$

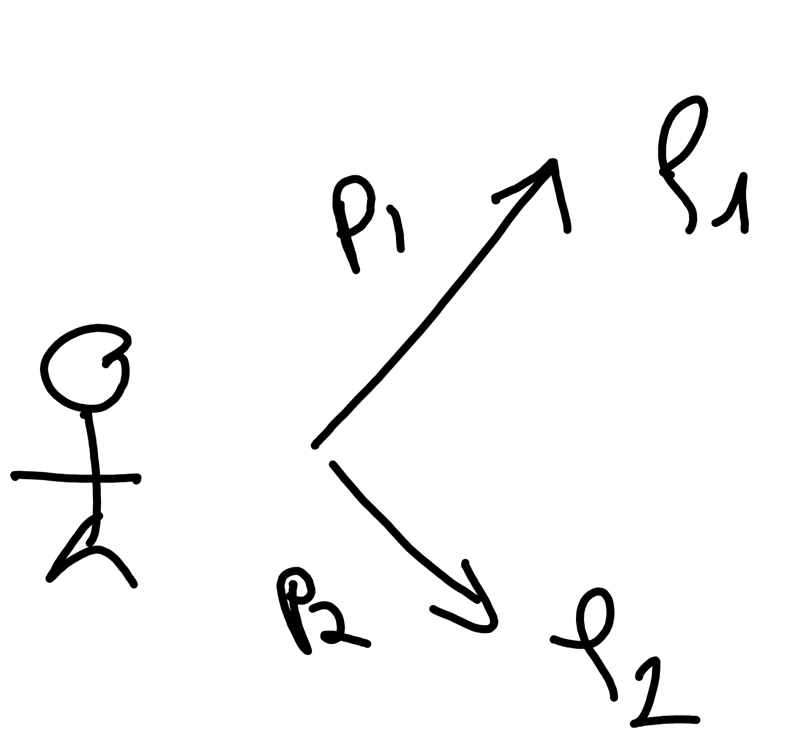
there is no measurement differentiating between them.

<u>Again:</u> given O, e.g. $M_i^\dagger M_i$ for a meas.,

$$\langle O\rangle = \frac{1}{2}\langle 0|O|0\rangle + \frac{1}{2}\langle 1|O|1\rangle = \mathrm{tr}\{\tfrac{1}{2}\mathbb{1}\cdot O\}.$$

The ensemble decomp. is not unique!

Important: probabilistic interpretation:



equivalent to

Thm $S(\mathcal{H})$ is convex: if $\rho_1, \rho_2 \in S(\mathcal{H})$ and $p \in [0,1]$, then $p\rho_1 + (1-p)\rho_2 \in S(\mathcal{H})$.

Proof (1) $\operatorname{tr}\{p\rho_1 + (1-p)\rho_2\} = p \cdot \operatorname{tr}\{\rho_1\} + (1-p)\operatorname{tr}(\rho_2)$

$= p + 1 - p = 1.$

(2) Given $|\psi\rangle \in \mathcal{H}$, $\langle\psi|(p_1\rho_1 + p_2\rho_2|\psi\rangle =$

$= p_1 \langle\psi|\rho_1|\psi\rangle + p_2\langle\psi|\rho_2|\psi\rangle \geq 0.$ □

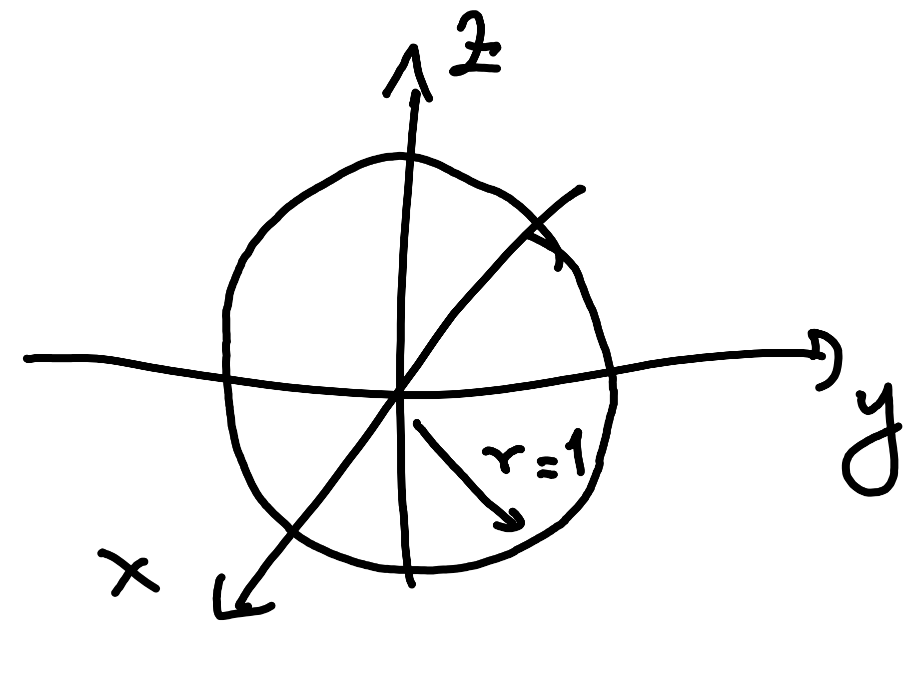
Example $\mathcal{H} = \mathbb{C}^2$. $\rho = \rho^\dagger$, $\operatorname{tr}\rho = 1$:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+c & a+ib \\ a-ib & 1-c \end{pmatrix} = \frac{1}{2}(1 + aX + bY + cZ)$$

When is ρ a density matrix?

Iff $\det\rho = \lambda_1 \cdot \lambda_2 = \lambda_1 \cdot (1-\lambda_1) \geq 0.$

$\det\rho = 1 - a^2 - b^2 - c^2 \geq 0$, so it's unit sphere:



$\rho = |\psi\rangle\langle\psi| \Leftrightarrow \det\rho = 0$

$\Leftrightarrow a^2 + b^2 + c^2 = 1,$

Bloch sphere.

- Pure states are on the outside
- Mixed states are on the inside.
- It's a convex set.

Higher dimensions: pure states are on the extremum, mixed states are inside.

Thm: Let $\rho \in S(\mathcal{H})$ be a pure state, i.e., assume $\exists |\psi\rangle \in \mathcal{H}$ s.t. $\rho = |\psi\rangle\langle\psi|$. Assume moreover that $\exists \rho_1, \rho_2 \in S(\mathcal{H})$, $p \in [0,1]$ s.t. $\rho = p\rho_1 + (1-p)\rho_2$.
Then ($p=0$ or $\rho_1 = \rho$) and ($p = 1$ or $\rho_2 = \rho$).

Proof: Take $|\varphi\rangle$ s.t. $\langle\psi|\varphi\rangle = 0$. Then

$$0 = \langle\varphi|\rho|\varphi\rangle = p_1\langle\varphi|\rho_1|\varphi\rangle + p_2\langle\varphi|\rho_2|\varphi\rangle$$

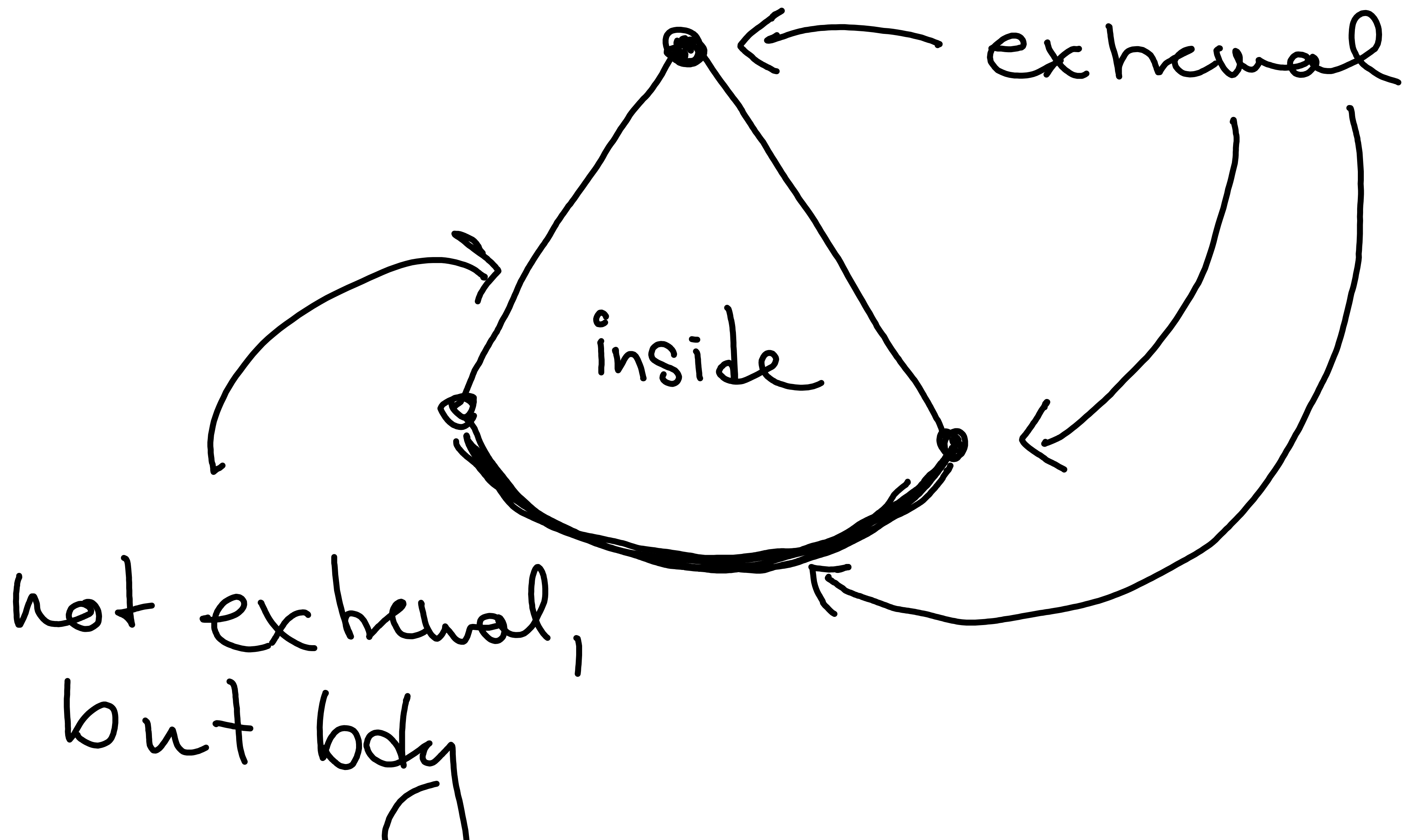
So either $p_1 = 0$ or $\langle\varphi|\rho_1|\varphi\rangle = 0 \ \forall \varphi \perp \psi$.

$\Rightarrow$ Last eig. vector is $|\psi\rangle$, $\rho_1 = |\psi\rangle\langle\psi|$.

Similarly for p_2 & ρ_2.

$\square$

Points like these are called extremal points of a convex set.



HW: prove that if $\rho \in S(\mathcal{H})$ is full rank, then it is in the inside of the set of density matrices: if $\eta \in S(\mathcal{H})$, then $\exists p \in (0,1)$ s.t.

$$p\rho + (1-p)\eta \in S(\mathcal{H}).$$

Remark: in $\mathbb{C}^3$, $\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ is not extremal, but on the bdy.

Summary: density matrix formalism:

- allows for probabilistic mixture (this is the convex str.)
- can describe "part" of a state.

Remark:
$$\rho_{AB} \to \rho_A = \mathrm{tr}_B\, \rho_{AB}, \qquad \rho_{AB} \to \rho_B = \mathrm{tr}_A\, \rho_{AB}$$
but in general can't go back.

(same as in proba theory, marginals)

Ex: $\frac{1}{2}\mathbb{1}_2$ is marginal both for $\frac{1}{4}\mathbb{1}_2 \otimes \mathbb{1}_2$ and $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

We introduced density matrices to be able to describe parts of a larger system; then we characterized as

$S(\mathcal{H}) = \{ \rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq 0, \ \mathrm{tr}\{\rho\} = 1$.

<u>Thm</u> $\forall \rho \in S(\mathcal{H})$ a f.d. Hilbert space $\mathcal{K}$, and

$|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$, $\|\psi\| = 1$ s.t.

$$\rho = \mathrm{tr}_K |\psi\rangle\langle\psi| .$$

<u>Proof</u>: Let us write $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$ ensemble decomp., e.g., eig. state decomp. $(p_i \geq 0)$

Let $\mathcal{K} = \mathbb{C}^n$ and $|\psi\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle \otimes |i\rangle$.

Then

$$\mathrm{tr}_B |\psi\rangle\langle\psi| = \mathrm{tr}_B \sum_{ij} \sqrt{p_i p_j}\, |\psi_i\rangle\langle\psi_j| \otimes |i\rangle\langle j|$$

$$= \sum_{ij} \sqrt{p_i p_j}\, |\psi_i\rangle\langle\psi_j| \cdot \underbrace{\mathrm{tr}\{|i\rangle\langle j|\}}_{\delta_{ij}} =$$

$$= \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho .$$

□