Time evolution / operations on density matrices

Unitary evolution: $|\psi\rangle \mapsto U|\psi\rangle$, then

$|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^+$.

Unitary evolution of density matrices:

$$\rho \longmapsto U\rho U^+.$$

Is this the most general thing we can do on a system? No: take a system, embed it in a larger one, time evolve there, trace out.

Example: $\rho \in S(\mathcal{H})$, add $\mathcal{K}$

(1) $\rho \mapsto \rho\otimes|0\rangle\langle 0|$

(2) $\rho\otimes|0\rangle\langle 0| \mapsto U(\rho\otimes|0\rangle\langle 0|)U^+$

(3) $tr_{\mathcal{K}}\left(U(\rho\otimes|0\rangle\langle 0|)U^+\right) =: T(\rho)$

Let us write $U = \sum_k A_k \otimes B_k$, then

$T(\rho) = tr_{\mathcal{K}}\left(U(\rho\otimes|0\rangle\langle 0|)U^+\right) =$

$$\sum_{kl} \mathrm{tr}_{\mathcal{K}} \left(A_k \rho A_l^{+} \otimes B_k |0\rangle\langle 0| B_l^{+} \right) =$$

$$= \sum_{kl} A_k \rho A_l^{+} \cdot \mathrm{tr}_{\mathcal{K}} \{ B_k |0\rangle\langle 0| B_l^{+} \} =$$

$$= \sum_{kl} A_k \rho A_l^{+} \cdot \sum_i \langle i| B_k |0\rangle\langle 0| B_l^{+} |i\rangle$$

$$= \sum_i \left(\sum_k A_k \cdot \langle i|B_k|0\rangle \right) \rho \left(\sum_l A_l^{+} \langle 0|B_l^{+}|i\rangle \right)$$

$$= \sum_i C_i \rho C_i .$$

So this kind of time evolution does:

$$\rho \longmapsto T(\rho) = \sum_i C_i \rho C_i^{+} .$$

Just as for density matrices, let us abstract away from the concrete time evolution and characterize this map by its properties:

We have considered the following time evolution: consider the state ρ, add another system to it (e.g. in the $|0\rangle\langle 0|$ state), time evolve w/ global unitary (i.e., unitary acting on both systems), then trace out the added system.

$$T(\rho) = \mathrm{tr}_B\left(\mathcal{U}\left(\rho \otimes |0\rangle\langle 0|\right)\mathcal{U}^\dagger\right).$$

We have seen:

$$T(\rho) = \sum_i C_i \rho C_i^\dagger,$$

with $C_i = (\mathbb{1} \otimes \langle i|)\,\mathcal{U}\,(\mathbb{1} \otimes |0\rangle)$.

<u>Def</u> (Kraus map): Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A linear map $T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is called a <u>Kraus map</u> if $\exists \{C_i\}_{i\in I} \subseteq \mathcal{L}in(\mathcal{H}, \mathcal{K})$ s.t.

$$T(\rho) = \sum_i C_i \rho C_i^\dagger.$$

<u>Example</u>: the above. Or even simpler,

$$\rho \mapsto U\rho U^\dagger \text{ is a Kraus map.}$$

The above example contained 2 more operations, each are Kraus:

(1) $\rho \mapsto \rho \otimes |0\rangle\langle 0|$, as

$$\rho \otimes |0\rangle\langle 0| = C\rho C^\dagger \text{ with } C \in \mathcal{L}in(\mathcal{H}, \mathcal{H}\otimes k)$$

$$C|\psi\rangle = |\psi\rangle \otimes |0\rangle.$$

(2) The partial trace. Here,

$$\mathrm{tr}_B(\rho_{AB}) = \sum_i C_i \rho_{AB} C_i^\dagger \text{ w/}$$

$$C_i \in \mathcal{L}in(\mathcal{H}\otimes k, \mathcal{H}), \; C_i|hk\rangle = |h\rangle \delta_{ik}.$$

<u>Remark</u>: composition of Kraus maps is Kraus.

(3) Post-measurement state (unnormalized!):

$$\rho \mapsto M_i \rho M_i^\dagger = p_i \rho_i$$

(4) <u>Expected</u> post-meas. state: $\rho = \sum_i p_i \rho_i = \sum_i M_i \rho M_i^\dagger$

<u>Remark</u>: Kraus maps form convex structure, probabilistic interpretation!

Thm: Let $T: B(\mathcal{H}) \to B(\mathcal{K})$ be Kraus.

Then $T(\rho) \geq 0$ if $B(\mathcal{H}) \ni \rho \geq 0$

Proof: We need to check expectation values of $T(\rho)$:

$$\langle \psi | T(\rho) | \psi \rangle = \sum_i \langle \psi | C_i \rho C_i^\dagger | \psi \rangle$$

$$= \sum_i \langle C_i^\dagger \psi | \rho | C_i^\dagger \psi \rangle \geq 0. \quad \square$$

We might have problem w/ trace preserving:

Thm Let $T: B(\mathcal{H}) \to B(\mathcal{K})$ be Kraus,

$$T(\rho) = \sum_i C_i \rho C_i^\dagger.$$

Then $\mathrm{tr}\{T(\rho)\} = \mathrm{tr}\{\rho\} \ \forall \rho \in B(\mathcal{H})$ iff

$$\sum_i C_i^\dagger C_i = \mathbb{1}.$$

Proof

$$\mathrm{tr}(T(\rho)) = \mathrm{tr}\left(\sum_i C_i \rho C_i^\dagger\right) = \mathrm{tr}\left\{\sum_i C_i^\dagger C_i \rho\right\},$$

and thus $\mathrm{tr}(T(\rho)\} = \mathrm{tr}\{\rho\} \ \forall \rho \in B(\mathcal{H})$ iff $\sum_i C_i^\dagger C_i = \mathbb{1}$. $\square$

Examples:

(1) $\rho \mapsto U\rho U^\dagger$ is trace-preserving

$$\mathrm{tr}\{U\rho U^\dagger\} = \mathrm{tr}\{U^\dagger U \rho\} = \mathrm{tr}\{\rho\}.$$

(2) $\rho \mapsto \rho \otimes |0\rangle\langle 0|$ is trace-preserving

(3) $\rho_{AB} \mapsto \rho_A = \mathrm{tr}_B \rho_{AB}$ — " —

(4) Given a measurement $\{M_i\}_{i=1}^{n}$, $\sum_i M_i^\dagger M_i = \mathbb{1}$,

the post-meas. constr.

$$\rho \mapsto M_i \rho M_i^\dagger$$

is not trace preserving. Can't make it trace preserving w/ normalization:

$$M_i^\dagger M_i \notin \mathbb{C} \cdot \mathbb{1}.$$

(i.e., the probability ($\mathrm{tr}\{M_i \rho M_i^\dagger\}$) depends on the state, there is no state-independent normalization).

Remark: composition of TP Kraus maps is TP.

HW Check $\sum_i C_i^\dagger C_i = \mathbb{1}$ for $C_i = (\mathrm{id} \otimes \langle i|) U (\mathrm{id} \otimes |0\rangle)$

We have seen that Kraus maps arise, for example, as $\rho \mapsto \rho\otimes|0\rangle\langle 0| \mapsto U(\rho\otimes|0\rangle\langle 0|)U^\dagger \mapsto \mathrm{tr}_B$.

Actually, all trace-preserving Kraus maps are like this.

<u>Thm:</u> Let $T: B(\mathcal{H}) \to B(\mathcal{H})$ be a Kraus map, $T(\rho) = \sum_{i=1}^{n} A_i \rho A_i^\dagger$. Then $\exists\, \mathcal{K}$ f.d. Hilbert space and $U \in B(\mathcal{H}\otimes\mathcal{K})$ unitary s.t.

$$T(\rho) = \mathrm{tr}_{\mathcal{K}}\, U\left(\rho\otimes|0\rangle\langle 0|\right)U^\dagger.$$

<u>Proof:</u> Let $K = \mathbb{C}^n$. Consider

$$X = \sum_i A_i \otimes |i\rangle\langle 0| = \begin{pmatrix} A_1 & 0 & & \\ A_2 & & \ddots & \\ \vdots & & & \\ A_n & & & 0 \end{pmatrix}$$

Then

$$X^\dagger X = \begin{pmatrix} \sum_i A_i^\dagger A_i & \\ & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \\ & 0 \end{pmatrix}$$

X has orthonormal columns. We can extend it to an orthonormal basis U:

$$U = \sum_{ij} U_{ij}\otimes|i\rangle\langle j|, \quad U_{i0} = A_i.$$

Then $\mathcal{U}(\rho \otimes |0\rangle\langle 0|)\mathcal{U}^+ =$

$$= \sum_i A_i \rho A_j^+ \otimes |i\rangle\langle j|$$

And thus $tr_B(\mathcal{U}(\rho \otimes |0\rangle\langle 0|)\mathcal{U}^+) = \sum_i A_i \rho A_i^+$.

$\square$

<u>HW</u>: what happens w/ Kraus maps of the form $B(\mathcal{H}) \to B(\mathcal{K})$?

Just as density matrices are pure states on a larger system, Kraus maps are unitaries on a larger system as well.

Can we have more general time evolution?

<u>Requirements</u>

- $\rho \longmapsto T(\rho)$ density matrix
- T s/b linear

Is this enough? No!

Take $T: A \mapsto A^T$. This is tr-preserving and positivity preserving (positive):

$$\langle\psi|A|\psi\rangle \geq 0 \;\forall\psi \iff \langle\psi|A^T|\psi\rangle \geq 0 \;\forall\psi$$

just by transposing.

Interesting if A and B share $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

or $\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Try: B applies the map T

$$(id \otimes T)(\rho) = \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^T & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^T \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and this in <u>not</u> a density matrix, as it is <u>not</u> positive.

<u>Takeaway</u>: even though T is positivity preserving, $id \otimes T$ (or $T \otimes id$) might not be.

Def: Let $T: B(\mathcal{H}) \to B(\mathcal{K})$ be linear. We say that

- T is positivity preserving, or simply positive, if $T(\rho) \geq 0$ if $\rho \geq 0$.
- T is n-positive if $T \otimes id : B(\mathcal{H}) \otimes B(\mathbb{C}^n) \to$
  $\to B(\mathcal{K}) \otimes B(\mathbb{C}^n)$
  is positive
- T is completely positive if it is n-positive $\forall n \in \mathbb{N}$.

The transposition map is not CP (completely positive).
Physical operations should be CP as one should able to manipulate entangled states as well.
Example CP map: unitary time evolution.
$T(\rho) = \mathcal{U} \rho \mathcal{U}^{+}$
Consider now $\rho \in \mathcal{H} \otimes \mathcal{K}$, $\rho \geq 0$, then
$(T \otimes id)(\rho) = (\mathcal{U} \otimes \mathbb{1}) \rho (\rho \otimes \mathbb{1})^{+} \geq 0$.

Similarly,

<u>Thm</u>: Every Kraus map is CP.

<u>Proof</u>: Same as for unitaries: if $\rho \in B(\mathcal{H})$,

$$T(\rho) = \sum_i C_i \rho C_i^\dagger \in B(\mathcal{K}),$$

then for $\rho \in B(\mathcal{H}) \otimes B(\mathcal{L})$

$$(T \otimes id_{B(\mathcal{L})})(\rho) = \sum_i (C_i \otimes \mathbb{1}_{\mathcal{L}}) \rho (C_i \otimes \mathbb{1}_{\mathcal{L}})^\dagger,$$

as $id_{B(\mathcal{L})}(X) = X = \mathbb{1}_{\mathcal{L}} X \mathbb{1}_{\mathcal{L}}$.

Then given $|\psi\rangle \in \mathcal{K} \otimes \mathcal{L}$,

$$\langle\psi|(T \otimes id)(\rho)|\psi\rangle = \sum_i \langle\psi| C_i \otimes \mathbb{1} \rho (C_i \otimes \mathbb{1})^\dagger |\psi\rangle \geq 0.$$

$\square$

So $\overset{TP}{\vee}$ Kraus maps describe physical operations!

We will show: every CP map is Kraus.

<u>Lemma</u> (see ex. class): Let $|\Omega\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

be the max. ent. state,

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle.$$ Then

$$(\mathbb{1} \otimes X)|\Omega\rangle = (X^T \otimes \mathbb{1})|\Omega\rangle \quad \forall X \in \mathcal{B}(\mathbb{C}^d).$$

Proof: direct calculation:

$$(\mathbb{1} \otimes X)|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{ijk} X_{ij} (\mathbb{1} \otimes |i\rangle\langle j|)|kk\rangle$$
$$= \frac{1}{\sqrt{d}} \sum_{ik} X_{ik} |ki\rangle.$$

$$(X^T \otimes \mathbb{1})|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{ijk} X_{ji} (|i\rangle\langle j| \otimes \mathbb{1}) |kk\rangle =$$
$$= \frac{1}{\sqrt{d}} \sum_{ik} X_{ki} |ik\rangle = \frac{1}{\sqrt{d}} \sum_{ik} X_{ik} |ki\rangle.$$

$\square$

— o —

Lemma: Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$. Then $\exists\, O \in \mathcal{B}(\mathcal{H})$ s.t. $|\psi\rangle = (O \otimes id)|\Omega\rangle$, $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$.

Proof: We can write $|\psi\rangle = \sum_i |\psi_i\rangle \otimes |i\rangle$, so if we define $O = \sum_k |\psi_k\rangle\langle k|$, then $(O \otimes id)|\Omega\rangle = |\psi\rangle$.

$\square$

General quantum operations: CP maps.

CP: $T : B(\mathcal{H}) \to B(\mathcal{K})$ s.t. $\forall \mathcal{L}$ and $\rho \in B(\mathcal{H}) \otimes B(\mathcal{L})$,

$$(T \otimes id_{B(\mathcal{L})})(\rho) \geq 0 \text{ if } \rho \geq 0.$$

Goal: to show that CP maps are Kraus:

$$T(\rho) = \sum_i C_i \rho C_i^\dagger$$

Note: we have seen that Kraus maps are CP.

Lemma (Last lecture): Let $|\Omega\rangle \in \mathcal{H} \otimes \mathcal{H}$ be the max. ent. state, $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$. Then $\forall O \in B(\mathcal{H})$

$$(O \otimes \mathbb{1})|\Omega\rangle = (\mathbb{1} \otimes O^T)|\Omega\rangle.$$

Lemma: Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$. Then $\exists\, O \in B(\mathcal{H})$ s.t. $|\psi\rangle = (O \otimes id)|\Omega\rangle$, $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$.

Proof: We can write $|\psi\rangle = \sum_i |\psi_i\rangle \otimes |i\rangle$, so if we define $O = \sum_k |\psi_k\rangle\langle k|$, then

$$(O \otimes id)|\Omega\rangle = |\psi\rangle.$$

$\square$

Thm: Let $T: B(\mathcal{H}) \to B(\mathcal{K})$ be linear. Then the following are equivalent:

(1) T is CP

(2) T is $\dim(\mathcal{H})$-positive

(3) $(id \otimes T)(|\Omega\rangle\langle\Omega|) \geq 0$, where $|\Omega\rangle \in \mathcal{H} \otimes \mathcal{H}$ is the max. ent. state, $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$

(4) T is Kraus.

Proof: We have seen $(4) \Rightarrow (1)$. $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ is by def. We show $(3) \Rightarrow (4)$.

Consider $|\Omega\rangle \in B(\mathcal{H}) \otimes B(\mathcal{H})$ max. ent. state $\left(|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle\right)$. We have seen $tr_B |\Omega\rangle\langle\Omega| = \mathbb{1}$.

So

$$T(\rho) = T(\rho \cdot \mathbb{1}) = T\left(\rho \; tr_A |\Omega\rangle\langle\Omega|\right) =$$

$$= (tr \otimes T)\left(\mathbb{1} \otimes \rho \; |\Omega\rangle\langle\Omega|\right) =$$

by Lemma

$$= (tr \otimes T)\left(\rho^T \otimes \mathbb{1} \; |\Omega\rangle\langle\Omega|\right)$$

$$= tr_A\left(\rho^T \otimes \mathbb{1} \cdot X\right), \text{ where}$$

$$X = (id \otimes T)(|\Omega\rangle\langle\Omega|) \underset{\text{by assumption}}{\geq} 0, \text{ and thus}$$

$X = \sum_k |\psi_k\rangle\langle\psi_k|$, e.g., $|\psi_k\rangle \in \mathcal{H}\otimes\mathcal{H}$ are unnormalized eigenvectors. But then we can write

$$X = \sum_k (C_k \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(C_k^+ \otimes \mathbb{1}),$$

so

$$T(\rho) = \mathrm{tr}_B\left[\sum_k (C_k \otimes \rho^T)|\Omega\rangle\langle\Omega|(C_k^+ \otimes \mathbb{1})\right]$$

$$= \mathrm{tr}_B\left[\sum_k (C_k \cdot \rho \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(C_k^+ \otimes \mathbb{1})\right]$$

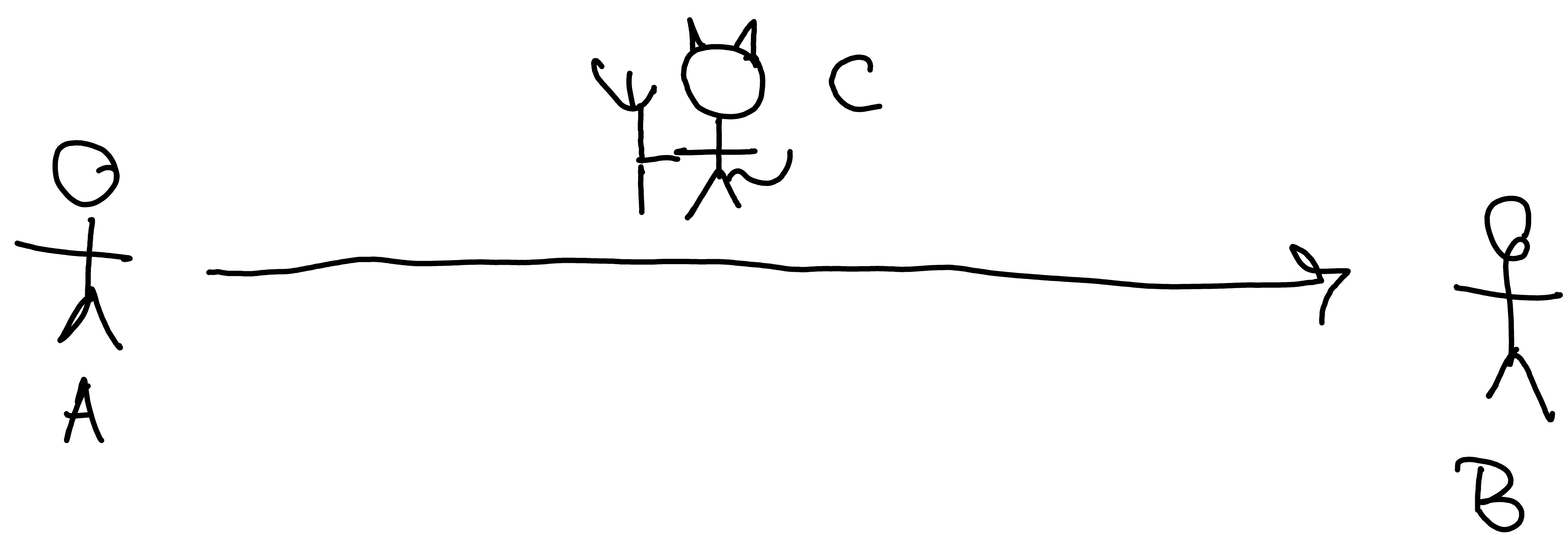
$$= \sum_k C_k \rho C_k^+. \qquad \square$$

<u>Remark:</u> We have seen that $(\mathrm{id}\otimes T)(|\Omega\rangle\langle\Omega|)$ is one-to-one. In fact,
$(\mathrm{id}\otimes T)(|\Omega\rangle\langle\Omega|) = \sum_{ij} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)$, it is easy to read off T from this mx. Can be used to check CP-ness! We have done so for transpose.

Significance of CP maps:

Can describe probabilistic time evolutions.

Typically, such scenario arises at communication tasks:



- A prepares a state $|\psi\rangle \in \mathcal{H}$
- Wants to send it to B.
- But the <u>channel</u> is noisy, $|\psi\rangle$ gets distorted.

<u>Example noise model</u>: $\mathcal{H} = \mathbb{C}^2$,

proba. p_0 — $|\psi\rangle$ gets transmitted properly

proba. p_x — $X|\psi\rangle$ gets transmitted instead of $|\psi\rangle$ — an X error occurs.

proba. p_y — $Y|\psi\rangle$ gets transmitted instead of $|\psi\rangle$

proba. p_2 - $Z|\psi\rangle$ gets transmitted.

We can then think of the message after transmission as the ensemble

$$\{(p_i, \sigma_i|\psi\rangle)\}_{i=0}^{3}.$$

The actual info transmitted is the corresponding density operator:

$$\rho = \sum_{i=0}^{3} p_i \cdot \sigma_i|\psi\rangle\langle\psi|\sigma_i^+,$$

instead of the density operator $|\psi\rangle\langle\psi|$.

That is, we model the channel by the CPTP map (<u>quantum channel</u>)

$$\rho \mapsto \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^+.$$

These maps are important not only in communication, but also at quantum computation: quantum computers are <u>noisy</u>, and this is how we model noise.