# Entanglement

* Want to understand the differences between classical & quantum.

* A <u>single</u> (low-dimensional) quantum system is not too exciting:
  - prepare a state } probability
  - time-evolve } distribution
  - measure

We can do the same just by tossing classical coins, i.e. by randomization + classical computation.

<u>BUT</u>! Usually one adds other requirements for such a simulation:

- For computation: you want to simulate a large (many qubit) system, but in limited time & memory.

This is hard: simulation of such system requires time/memory.

- In quantum info/communication you want to simulate s.t. two parts of your system are spatially separated, and the operations you make on the two parts can only be synchronized by classical communication.

In general: composite quantum systems are more complex than just a classical mixture of quantum systems. This is what we call entanglement.

<u>Def</u> (Entanglement, pure states): A state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is <u>separable</u> iff it is of the form $|\psi\rangle = |\varphi\rangle \otimes |\chi\rangle$,

$|\varphi\rangle \in \mathcal{H}_A$, $|\chi\rangle \in \mathcal{H}_B$. The state $|\psi\rangle$ is <u>entangled</u> if it is not separable.

We have seen that $|\Omega\rangle = \frac{1}{2}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is entangled.

For mixed states (density matrices), the definition has to be compatible w/ the convex structure.

<u>Def</u> (Entanglement of mixed states):
A state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ is <u>separable</u> if it can be written as

$$\rho = \sum_i p_i \, \nu_i \otimes \eta_i,$$

where $p_i \geq 0$, $\sum_i p_i = 1$, $\nu_i \in S(\mathcal{H}_A)$, $\eta_i \in S(\mathcal{H}_B)$.

<u>Sanity check</u>: the two definitions are compatible. For that, let $\rho$ be a density matrix that is separable <u>and</u> describes a pure state.

* separable: $\rho = \sum_i p_i \, \nu_i \otimes \eta_i$

* pure : $\rho = \sum_i p_i \, \rho_i$ convex comb. of density matrices $\Rightarrow$ the convex comb. is trivial

Thus a density matrix describing a pure separable state is of the form

$$\rho = \nu \otimes \eta .$$

But $\rho$ is rank-1 iff both $\nu$ and $\eta$ are rank 1, and thus it is of the form

$$\rho = (|\psi_1\rangle \otimes |\psi_2\rangle)(\langle\psi_1| \otimes \langle\psi_2|).$$

Given a state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$, how to detect whether $\rho$ is entangled?

One possibility:

<u>Prop</u>: Let $T : B(\mathcal{H}_B) \to B(\mathcal{K})$ be a positive map, $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a <u>separable</u> state. Then $(id \otimes T)(\rho) \geq 0$.

Proof: if $\rho = \sum_i p_i \, \eta_i \otimes \nu_i$ w/ $p_i \geq 0$, $\eta_i \geq 0$, $\nu_i \geq 0$, then

$$\rho = \sum_i p_i \, \eta_i \otimes T(\nu_i) \geq 0$$

as well as $T(\nu_i) \geq 0$. $\square$

This can be used to detect entanglement: if $\rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a state, $T : B(\mathcal{H}_B) \to B(\mathcal{K})$ a positive map and $(id \otimes T)(\rho) \ngeq 0$, then $T$ is not CP AND $\rho$ is entangled.

Example: if $T$ is transposition, then

$$(id \otimes T)(|\Omega\rangle\langle\Omega|) \ngeq 0 \Rightarrow |\Omega\rangle \text{ is entangled.}$$

In gen, this is an easily checkable criterion: write $\rho = \sum_{ij} |i\rangle\langle j| \otimes \rho_{ij}$, Then if $(id \otimes T)(\rho) = \sum_{ij} |i\rangle\langle j| \otimes \rho_{ij}^T \ngeq 0$, then $\rho$ is entangled.

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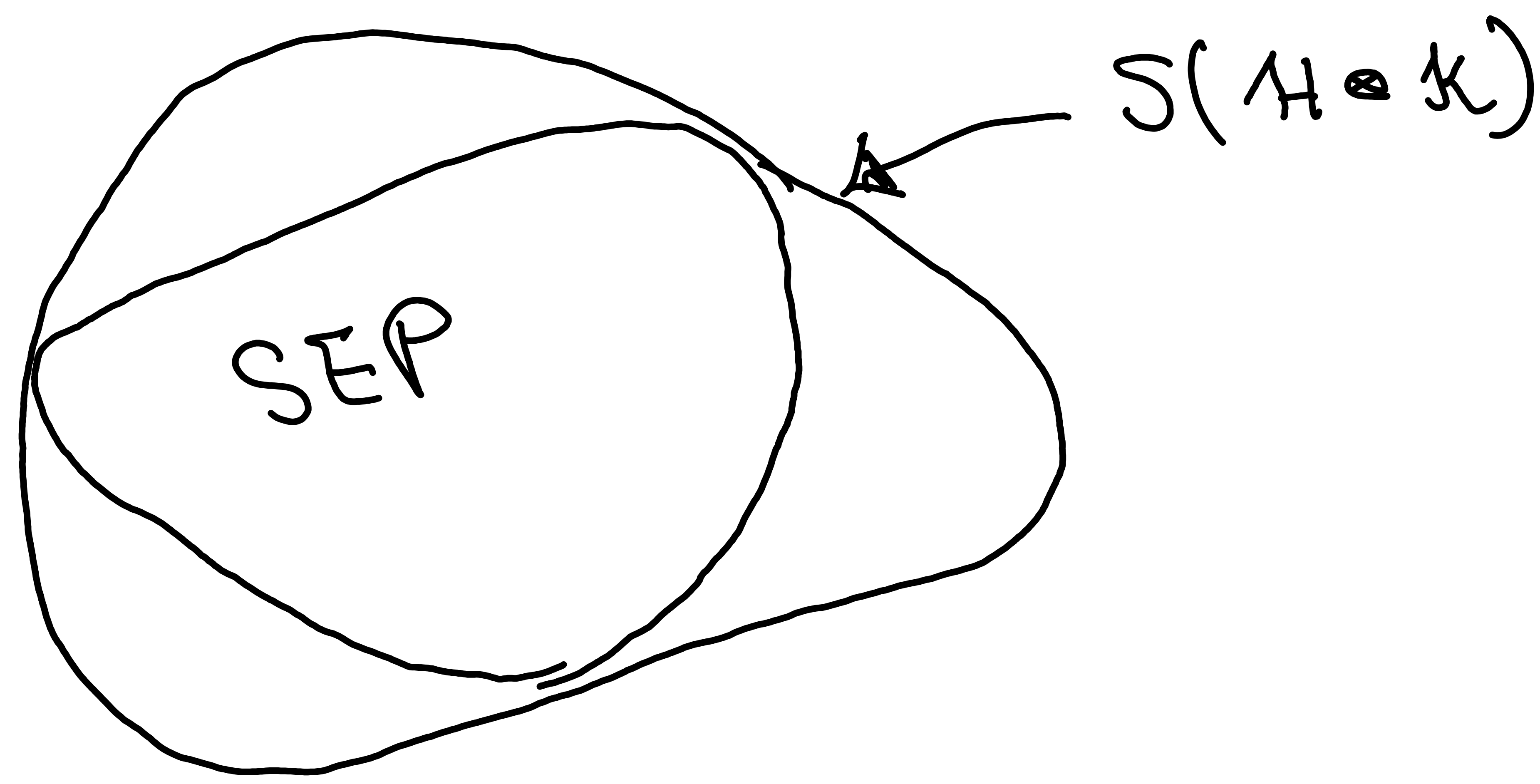
# Entanglement

$\rho \in S(\mathcal{H} \otimes \mathcal{K})$ is separable if $\rho$ can be written as $\rho = \sum_i p_i \eta_i \otimes \nu_i$, where $p_i \in \mathbb{C}$, $\eta_i \in B(\mathcal{H})$ and $\nu_i \in B(\mathcal{K})$ are s.t. $p_i \geq 0$ (so also real), $\eta_i \in S(\mathcal{H})$, $\nu_i \in S(\mathcal{K})$.

Note that this is a convex set,




$\rho \in S(\mathcal{H} \otimes \mathcal{K})$ is <u>entangled</u> if it is not separable.

Convex structure = probabilistic mixture, this is purely classical. So in this sense, entanglement is everything in the composite system that can't be explained by classical operations. We will make this notion more precise.

Remark: for pure states, separable = product, entangled = not product.

We have seen a way to detect entanglement: take a positive but not CP map, then if $(\mathrm{id} \otimes T)(\rho) \not\geq 0$, then $\rho$ is entangled.

Example: $T$ = transposition; this way of ent. detection is called PPT = positive partial transpose criterion.

Remark: one can prove that in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ this detects all entanglement.

Example #2 $T(\rho) = \mathrm{tr}(\rho) \cdot \mathbb{1} - \rho$. This is positive as $\rho \leq \lambda_{max} \cdot \mathbb{1} \leq \mathrm{tr}(\rho) \mathbb{1}$. Applying it to a composite system:

$$(\mathrm{id} \otimes T)(\rho_{AB}) = \rho_A \otimes \mathbb{1} - \rho_{AB}.$$

This is called <u>reduction criterion</u>: if

$\rho_A \otimes \mathbb{1} - \rho_{AB} \not\geq 0$, then $\rho_{AB}$ is entangled.

For $\rho_{AB} = |\Omega\rangle\langle\Omega|$, $\rho_A = \frac{1}{2}\mathbb{1}_2$, and

$$\rho_A \otimes \mathbb{1} - \rho_{AB} = \frac{1}{2}\begin{pmatrix} 0 & & & -1 \\ & 1 & & \\ & & 1 & \\ -1 & & & 0 \end{pmatrix} \not\geq 0.$$

<u>Note</u>: $T(\rho) = \mathrm{tr}(\rho)\cdot\mathbb{1} - \rho$ is <u>not</u> TP.

This method of detection works if you know the density matrix. One can do a more practical approach: through measurement.

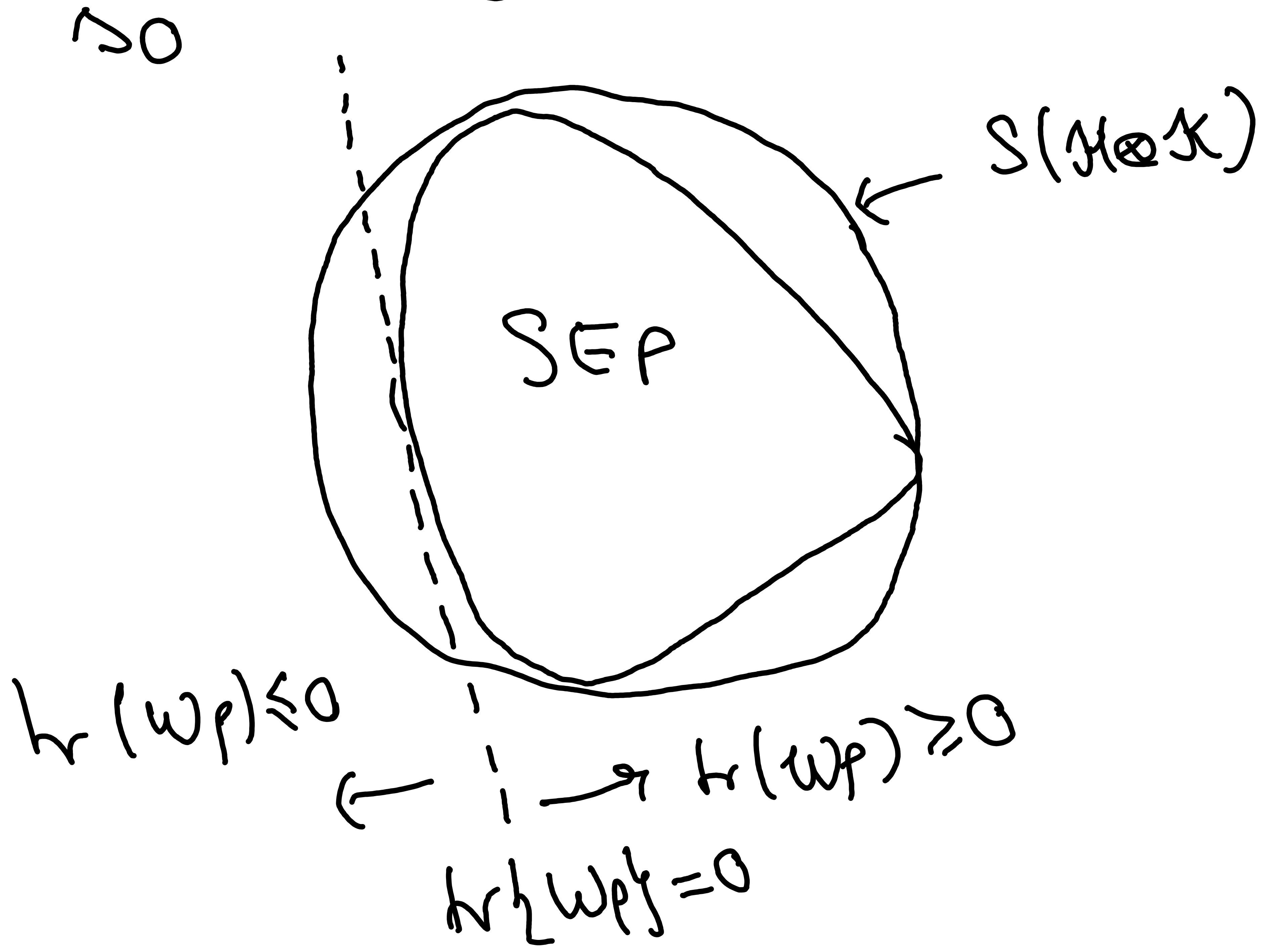
<u>Definition</u> $W \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ is an <u>entanglement witness</u> if $W = W^{\dagger}$, it is <u>not</u> positive semidefinite and $\mathrm{tr}\{\rho W\} \geq 0$ $\forall \rho$ separable in $\mathcal{S}(\mathcal{H}\otimes\mathcal{K})$.

This $W$ then encodes a measurement: write $W = \sum_\lambda \lambda \cdot P_\lambda$, its spectral decomposition, then $\{P_\lambda\}_{\lambda\in\mathrm{Spec}(W)}$ define the measurement.

if we measure $\rho$ (you need to prepare $\rho$ many times to form statistics), and get output exp. value $\leq 0$, then $\rho$ is entangled.

As $W$ is <u>not</u> positive, it detects some entanglement: $\exists |\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ s.t. $\langle\psi|W|\psi\rangle \ngeq 0$, $|\psi\rangle$ is entangled.

Pictorially, $tr\{\rho W\} = 0$ is a hyperplane, so




How to construct ent. witnesses? Through the previous method!

**Thm**: Let $T: B(\mathcal{H}) \longrightarrow B(\mathcal{K})$ be positive but not CP map. Then $W \in B(\mathcal{H} \otimes \mathcal{K})$,

$$W = (\mathrm{id} \otimes T)\left( |\Omega\rangle\langle\Omega| \right),$$

where $|\Omega\rangle \in \mathcal{H} \otimes \mathcal{H}$ is the max. ent. state, is an entanglement witness.

**Proof**: $T$ is positive and thus $(T \otimes \mathrm{id})$ is Hermitian preserving $\Rightarrow W = W^{+}$.

$T$ is not CP, so $W$ is not positive semidefinit. Finally, if $\rho \in S(\mathcal{H} \otimes \mathcal{K})$ is separable, then $\rho = \sum_i p_i \nu_i \otimes \eta_i$ w/ $p_i \geq 0$, $\nu_i \geq 0$ and $\eta_i \geq 0$, so

$$\mathrm{tr}(\rho W) = \sum_i p_i \, \mathrm{tr}\left\{ \nu_i \otimes \eta_i \cdot (\mathrm{id} \otimes T)(|\Omega\rangle\langle\Omega|) \right\} =$$

$$\overset{*}{=} \sum_i p_i \, \mathrm{tr}\left\{ (\mathbb{1} \otimes \eta_i) \cdot (\mathrm{id} \otimes T)\left( (\nu_i \otimes \mathbb{1}) |\Omega\rangle\langle\Omega| \right) \right\}$$

$$= \sum_i p_i \, \mathrm{tr}\left\{ (\mathbb{1} \otimes \eta_i) \cdot (\mathrm{id} \otimes T)\left( (\mathbb{1} \otimes \nu_i^T) |\Omega\rangle\langle\Omega| \right) \right\}$$

$$= \sum_i p_i \frac{1}{d} \sum_{kl} \mathrm{tr}\left\{ (\mathbb{1} \otimes \eta_i) \cdot (\mathrm{id} \otimes T)\left( \mathbb{1} \otimes \nu_i^T |kk\rangle\langle ll| \right) \right\}$$

$$= \sum_i p_i \frac{1}{d} \sum_{kl} \mathrm{tr}\left\{ (\mathbb{1}\otimes\eta_i)(\mathrm{id}\otimes T)\left( |k\rangle\langle l| \otimes \nu_i^T |k\rangle\langle l| \right)\right\}$$

$$= \sum_i p_i \frac{1}{d} \sum_{kl} \mathrm{tr}\left\{ |k\rangle\langle l| \otimes \eta_i T\left(\nu_i^T |k\rangle\langle l|\right)\right\}$$

$$= \sum_i p_i \frac{1}{d} \sum_{kl} \underbrace{\mathrm{tr}\left\{|k\rangle\langle l|\right\}}_{\delta_{kl}} \cdot \mathrm{tr}\left\{ \eta_i T\left(\nu_i^T |k\rangle\langle l|\right)\right\}$$

$$= \sum_i p_i \frac{1}{d} \sum_{l} \mathrm{tr}\left\{ \eta_i T\left(\nu_i^T |l\rangle\langle l|\right)\right\}$$

$$= \sum_i p_i \frac{1}{d} \sum_{l} \mathrm{tr}\left\{ \eta_i T\left(\nu_i^T\right)\right\}.$$

Here $\nu_i^T \geq 0$, and $T$ is positivity preserving, so $T(\nu_i^T) \geq 0$. $\eta_i \geq 0$ as well, and thus $\mathrm{tr}\{\eta_i T(\nu_i^T)\} = \mathrm{tr}\{\eta_i^{1/2} T(\nu_i^T)\,\eta_i^{1/2}\} \geq 0$, as $\eta_i^{1/2} T(\nu_i^T)\,\eta_i^{1/2} \geq 0$.

So if $\rho$ is separable, then $T(\rho W) \geq 0$.

* We have used $(O\otimes\mathbb{1})|\Omega\rangle = (\mathbb{1}\otimes O^T)|\Omega\rangle$.

$\square$

We can thus detect entanglement through suitable measurements.

Actually, for every entangled state we can create a measurement that detects the entanglement of that given state. $\boxed{HW}$