Quantum teleportation & dense coding

Assume A has an (unknown) state $|\chi\rangle \in \mathbb{C}^2$.
Her goal is to transfer this state to B, but only with classical communication.

If she had many copies of $|\chi\rangle$, she could measure many times, gather statistics, and thus obtain the exact form $|\chi\rangle = \chi_0|0\rangle + \chi_1|1\rangle$, (also known as tomography)
then communicate χ_0, χ_1 to Bob, who can then prepare the state.

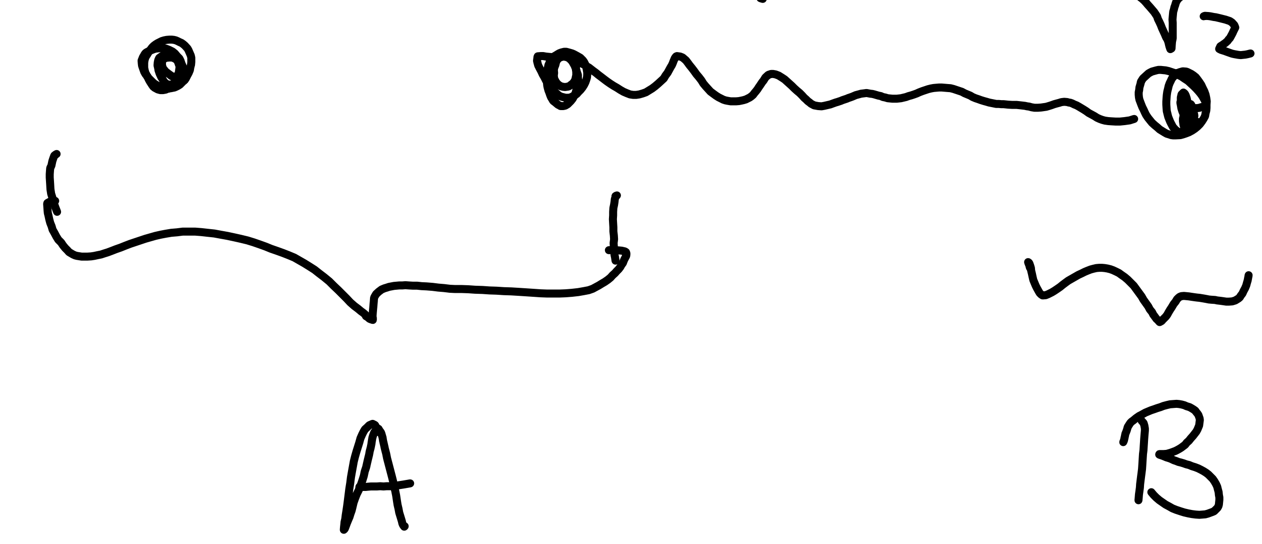
What if she has only a single copy of $|\chi\rangle$?

Problem: measurement destroys the state, she can't obtain classical description.

However, if A and B share an

entangled state, then A can still transfer $|\chi\rangle$ to B. Only 2 bits are needed!

<u>Setup:</u> $|\chi\rangle$ $\quad |\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$



<u>To notice</u>: $\mathbb{C}^2 \otimes \mathbb{C}^2$ has the following basis:

$$\left.\begin{aligned}
|\Omega_{00}\rangle &= |\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\\
|\Omega_{01}\rangle &= (\mathbb{1} \otimes Z)|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)\\
|\Omega_{10}\rangle &\; (\mathbb{1} \otimes X)|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)\\
|\Omega_{11}\rangle &= (\mathbb{1} \otimes XZ)|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)
\end{aligned}\right\} \; |\Omega_{\alpha\beta}\rangle = X^{\alpha} Z^{\beta} \cdot |\Omega\rangle$$

We can thus write $|\chi\rangle \otimes |\Omega\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\chi\rangle \otimes |\Omega\rangle = \sum_{\alpha\beta} |\Omega_{\alpha\beta}\rangle \otimes |\psi_{\alpha\beta}\rangle,$$ for some $|\psi_{\alpha\beta}\rangle$. How to calculate $|\psi_{\alpha\beta}\rangle$?

(a) Brute force

(b) Notice that

$$|\Psi_{\alpha\beta}\rangle = \left(\langle\Omega_{\alpha\beta}|\otimes id\right)\left(|x\rangle\otimes|\Omega\rangle\right)$$

$$= \left(\langle\Omega|\otimes id\right)\left(id\otimes Z^{\beta}X^{\alpha}\otimes id\right)\left(|x\rangle\otimes|\Omega\rangle\right)$$

$$= \left(\langle\Omega|\otimes id\right)\left(id\otimes id\otimes X^{\alpha}Z^{\beta}\right)\left(|x\rangle\otimes|\Omega\rangle\right)$$

$$(X\otimes id)|\Omega\rangle = (id\otimes X)|\Omega\rangle$$

$$(Z\otimes id)|\Omega\rangle = (id\otimes Z)|\Omega\rangle$$

$$= X^{\alpha}Z^{\beta}\cdot\underbrace{\left(\langle\Omega|\otimes id\right)\left(id\otimes|\Omega\rangle\right)}\,|x\rangle$$

$*$

$$\left(\langle\Omega|\otimes id\right)\left(id\otimes|\Omega\rangle\right) = \frac{1}{2}\left(\langle 00|\otimes id + \langle 11|\otimes id\right)\cdot$$

$$\cdot\left(id\otimes|00\rangle + id\otimes|11\rangle\right) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right),$$

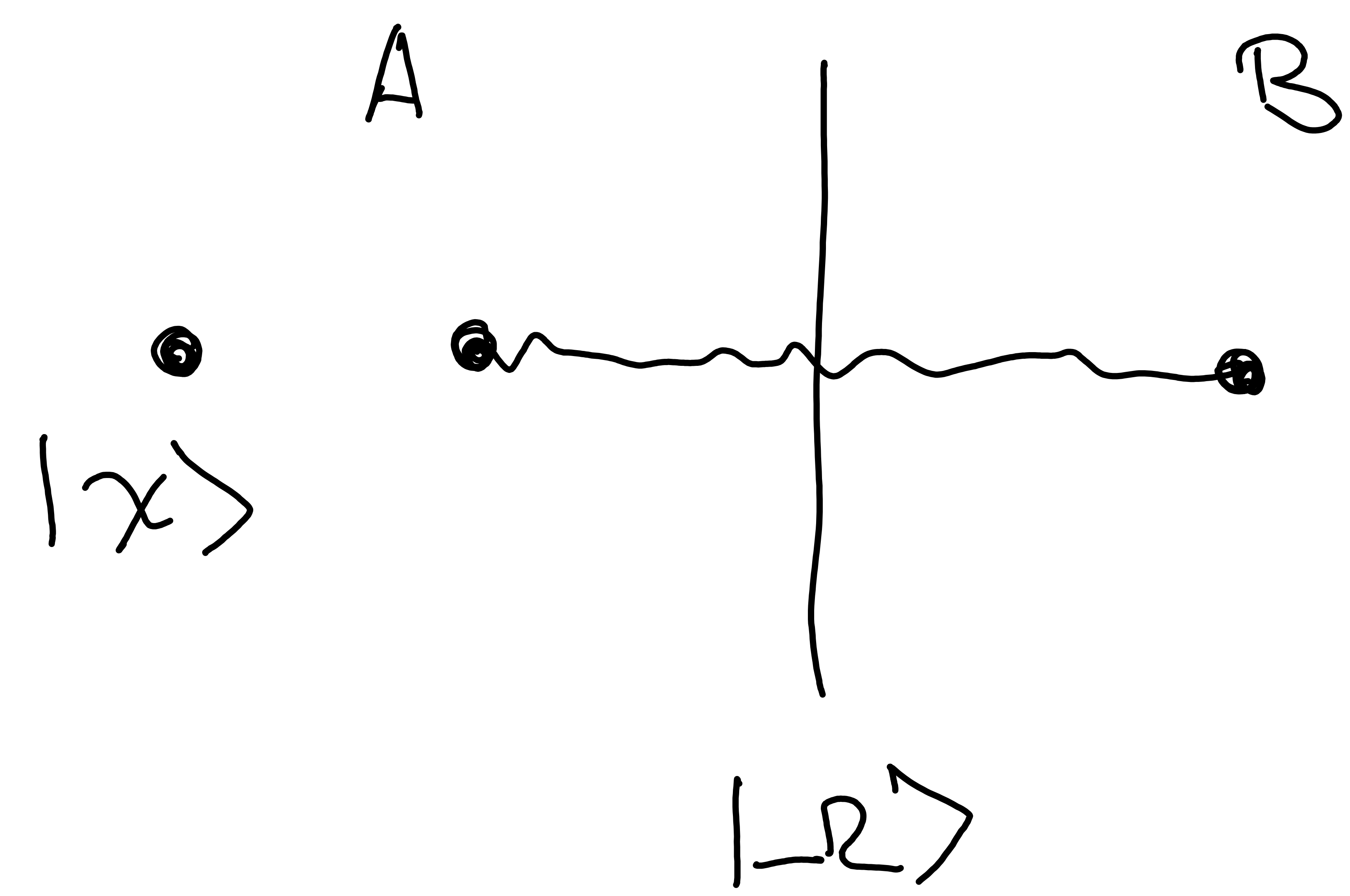
So this is just the identity!

We thus get that

$$|x\rangle\otimes|\Omega\rangle = \sum_{\alpha\beta}|\Omega_{\alpha\beta}\rangle\otimes X^{\alpha}Z^{\beta}|x\rangle.$$

Missing: $*$

Teleportation:



3-partite system: $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$,

$$|\psi_{ABC}\rangle = \underset{\mathcal{H}_{A_1} \cong \mathbb{C}^2}{\underset{\cap}{|\chi\rangle}} \otimes \underset{\mathcal{H}_{A_2} \otimes \mathcal{H}_B \cong \mathbb{C}^2 \otimes \mathbb{C}^2}{\underset{\cap}{|\Omega\rangle}}$$

<u>Goal</u>: transfer $|\chi\rangle$ to B's side.

<u>Rules</u>: A can manipulate her side (A_1 & A_2), B his side (B), but they can't act globally. However, they are allowed to communicate.

Can they achieve their goal? Yes!

For that, let us rewrite the state w.r.t. the $A_1A_2 | B$ bipartition. Actually, we want a specific ONB on A_1A_2:

$$|\Omega_{\alpha\beta}\rangle = (1 \otimes X^{\alpha} Z^{\beta})|\Omega\rangle$$

This is called the Bell basis.

$$|\Omega_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Omega_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Omega_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Omega_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

So we want $|\chi_{\alpha\beta}\rangle$ s.t.

$$|\Psi_{A_1A_2B}\rangle = |\chi\rangle \otimes |\Omega\rangle = \sum_{\alpha\beta} |\Omega_{\alpha\beta}\rangle \otimes |\chi_{\alpha\beta}\rangle.$$

As $|\Omega_{\alpha\beta}\rangle$ form an ONB,

$$|\chi_{\alpha\beta}\rangle = \underbrace{\Big(\underbrace{\langle\Omega_{\alpha\beta}|}_{\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2} \to \mathbb{C}} \otimes \mathrm{id}\Big)}_{\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\mathcal{H}_B \to \mathcal{H}_B} \underbrace{|\Psi_{A_1A_2B}\rangle}_{\in\, \mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\mathcal{H}_B}$$

(Also could be noted as $\langle\Omega_{\alpha\beta}|_{A_1A_2} |\Psi_{A_1A_2B}\rangle$

— here the index $A_1 A_2$ means that the lin. fcn. $\langle \Omega_{\alpha\beta}| : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}$ acts on the $A_1 A_2$ component)

<u>So</u>

$$|\chi_{\alpha\beta}\rangle = (\langle \Omega| \otimes \mathbb{1})(\mathbb{1} \otimes Z^\beta X^\alpha \otimes \mathbb{1})(|\chi\rangle \otimes |\Omega\rangle)$$

$$\underbrace{\langle \Omega_{\alpha\beta}| = (|\Omega_{\alpha\beta}\rangle)^\dagger}_{\langle \Omega_{\alpha\beta}| \otimes \mathbb{1}} = ((\mathbb{1} \otimes X^\alpha Z^\beta)|\Omega\rangle)^\dagger$$

Using now $(O \otimes \mathbb{1})|\Omega\rangle = (\mathbb{1} \otimes O^T)|\Omega\rangle$, with $O = Z^\beta X^\alpha$, we obtain

$$|\chi_{\alpha\beta}\rangle = (\langle \Omega| \otimes X^\alpha Z^\beta)(|\chi\rangle \otimes |\Omega\rangle)$$
$$= X^\alpha Z^\beta \cdot |\chi_{oc}\rangle$$

Here

$$|\chi_{oc}\rangle = (\langle \Omega| \otimes \mathbb{1})(|\chi\rangle \otimes |\Omega\rangle)$$
$$= \frac{1}{2} \sum_{ij} (\langle i| \otimes \langle i| \otimes \mathbb{1})(|\chi\rangle \otimes |j\rangle \otimes |j\rangle)$$
$$= \frac{1}{2} \sum_{ij} \langle i|\chi\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} \cdot |j\rangle = \frac{1}{2}|\chi\rangle.$$

We can thus write

$$|\chi\rangle \otimes |\Omega\rangle = \frac{1}{2}\sum_{\alpha\beta} |\Omega_{\alpha\beta}\rangle \otimes X^{\alpha}Z^{\beta}|\chi\rangle .$$

So we can do the following to transfer $|\chi\rangle$ to Bob's side:

1. A measures her 2 qubits in the $|\Omega_{\alpha\beta}\rangle$ basis (i.e., w/ meas. operators $\{|\Omega_{\alpha\beta}\rangle\langle\Omega_{\alpha\beta}|\}_{\alpha,\beta\in\{0,1\}}$).
   For outcome $\alpha\beta$, the post-meas. state is $|\Omega_{\alpha\beta}\rangle \otimes X^{\alpha}Z^{\beta}|\chi\rangle$.

2. She communicates the outcome α and β to Bob.

3. Bob knows that after measurement he has the state $X^{\alpha}Z^{\beta}|\chi\rangle$, so applies $Z^{\beta}X^{\alpha}$ to get back $|\chi\rangle$.

We can further break down "A measures in the $|\Omega_{\alpha\beta}\rangle$ basis":

(1) Find $\mathcal{U}$ s.t. $\mathcal{U}|\alpha\beta\rangle = |\Omega_{\alpha\beta}\rangle$

(2) Instead of measuring w/ $|\Omega_{\alpha\beta}\rangle$, first apply $\mathcal{U}^\dagger$, then measure $|\alpha\beta\rangle$, i.e. comp. basis in both qubits, then rotate back w/ $\mathcal{U}$. We don't really need A's part of the state, so we can skip this part.

<u>Note</u>: if $Q_i = \mathcal{U}P_i\mathcal{U}^\dagger$, then

* $\mathrm{tr}\{Q_i\rho\} = \mathrm{tr}\{P_i\mathcal{U}^\dagger\rho\mathcal{U}\}$
* $Q_i\rho Q_i^\dagger = \mathcal{U}P_i\mathcal{U}^\dagger\rho\,\mathcal{U}P_i^\dagger\mathcal{U}^\dagger$.

Finally note that

$$|\Omega_{\alpha\beta}\rangle = \mathrm{CNOT}(H\otimes\mathbb{1})|\alpha\beta\rangle :$$

$\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, $HX = ZH$

$|0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes X$

$$CNOT(H\otimes 1)|\beta\alpha\rangle = CNOT(H\otimes 1)(X^\beta\otimes X^\alpha)|00\rangle$$

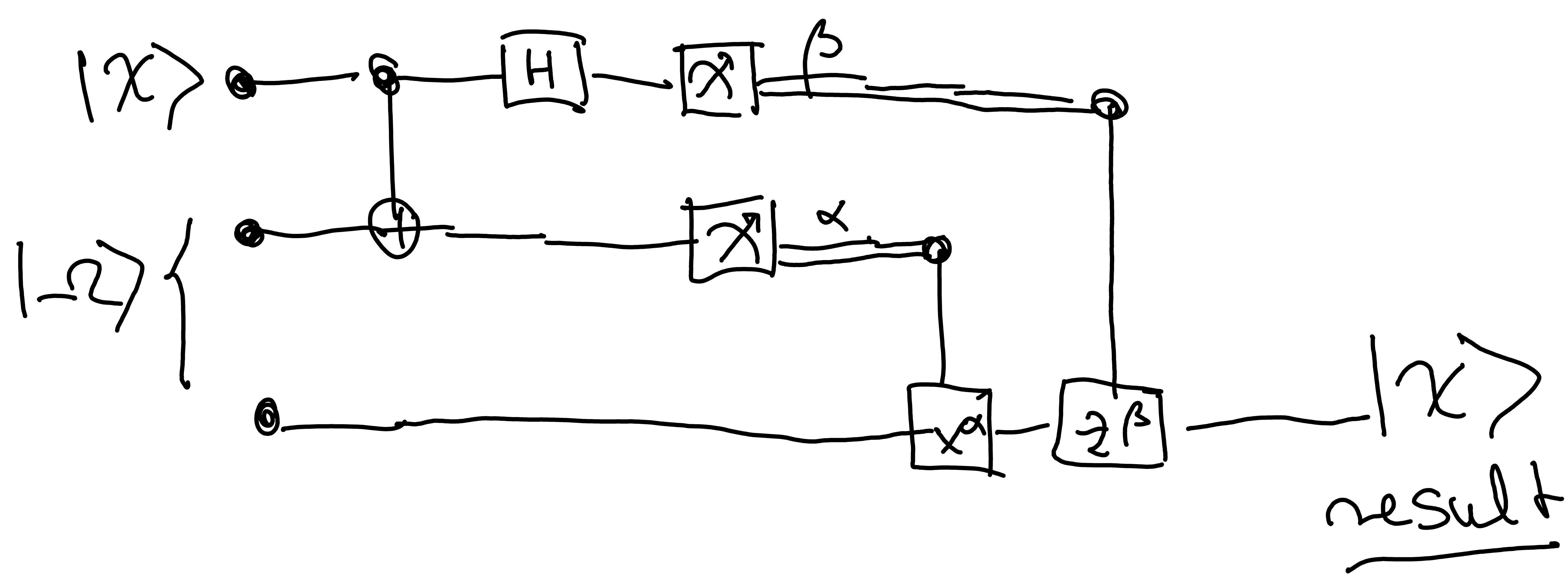
$$= CNOT\,(Z^\beta\otimes X^\alpha)(H\otimes 1)|00\rangle$$

$$= (Z^\beta\otimes X^\alpha)\,\underbrace{CNOT\,\underbrace{(H\otimes 1)|00\rangle}_{\frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)}}_{\frac{1}{\sqrt{2}}|00\rangle+|11\rangle \,=\, |\Omega\rangle}$$

$$= (1\otimes X^\alpha Z^\beta)|\Omega\rangle.$$

So if we first undo CNOT, then $(H\otimes 1)$, we can measure in the comp. bases!

<u>Final protocol:</u>



Notice: We did Local Operations assisted with Classical Communication, LOCC.

Such operations should not increase entanglement, as they are essentially classical.

In our case, we have started from

$$|\chi\rangle \otimes |\Omega\rangle \in (\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \otimes \mathcal{H}_B,$$

which is entangled (wrt. A-B cut!)

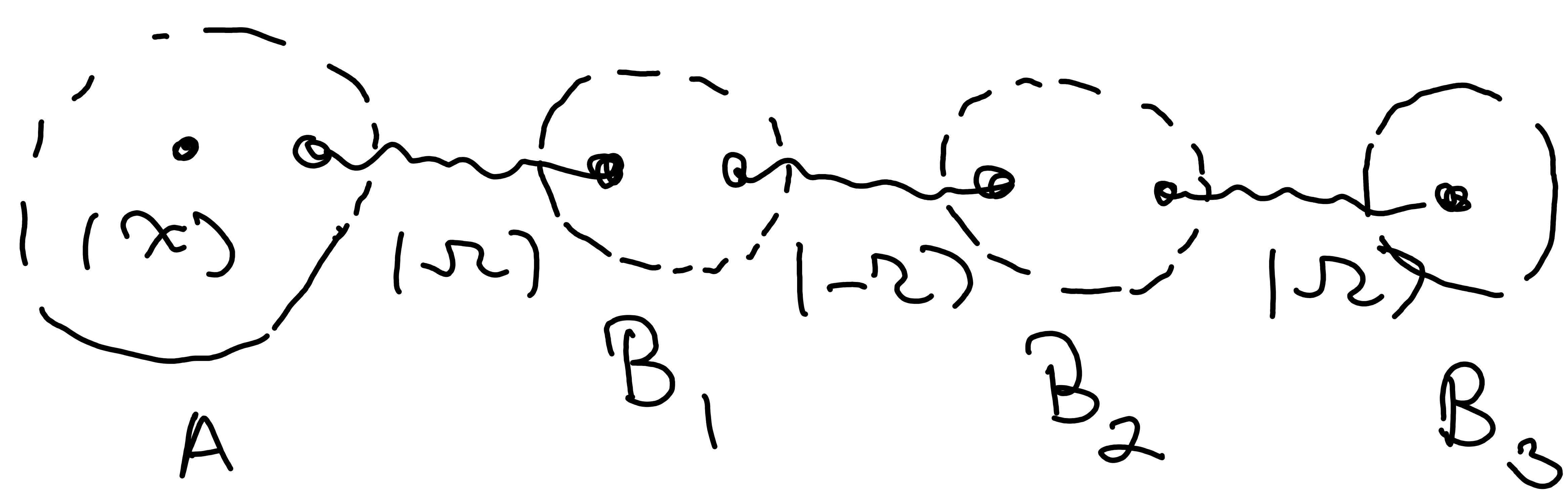
After the operation, if we measure $\alpha\beta$, we end up in $|\alpha\beta\rangle \otimes |\chi\rangle$.

Actually, we can create $U_{\alpha\beta} \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ s.t. we always end up in the same state, e.g. $|00\rangle \otimes |\chi\rangle$.

This state is NOT entangled wrt A|B cut,

LOCC can destroy entanglement.

<u>Remark:</u> teleportation can be used to transfer a state to large distances



We can transfer $|\chi\rangle$ to B_3 by repeated teleportation! This is called quantum repeater.

<u>Remark II</u>: Observe how similar this is to the Choi-Jamiołkowski isomorphism (when we get the Ω outcome): there also we "teleport ρ" through the max. ent. state.

Dense coding

If A & B share an entangled state, then A can send B two classical bit of info via sending a single qubit.

$\boxed{1}$ A & B share $|\Omega\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

$\boxed{2}$ A prepares $|\Omega_{\alpha\beta}\rangle = (Z^\beta X^\alpha \otimes \mathbb{1})|\Omega\rangle$

$\boxed{3}$ After sending A's part to Bob, Bob can measure $|\Omega_{\alpha\beta}\rangle$, so by sending 1 qubit, A could send 2 classical bit information.

Remark: Needed a pre-shared max. ent. state, so all together 2 qubit transfer is needed. But 1 qubit can be transferred before the message.