Bell inequalities

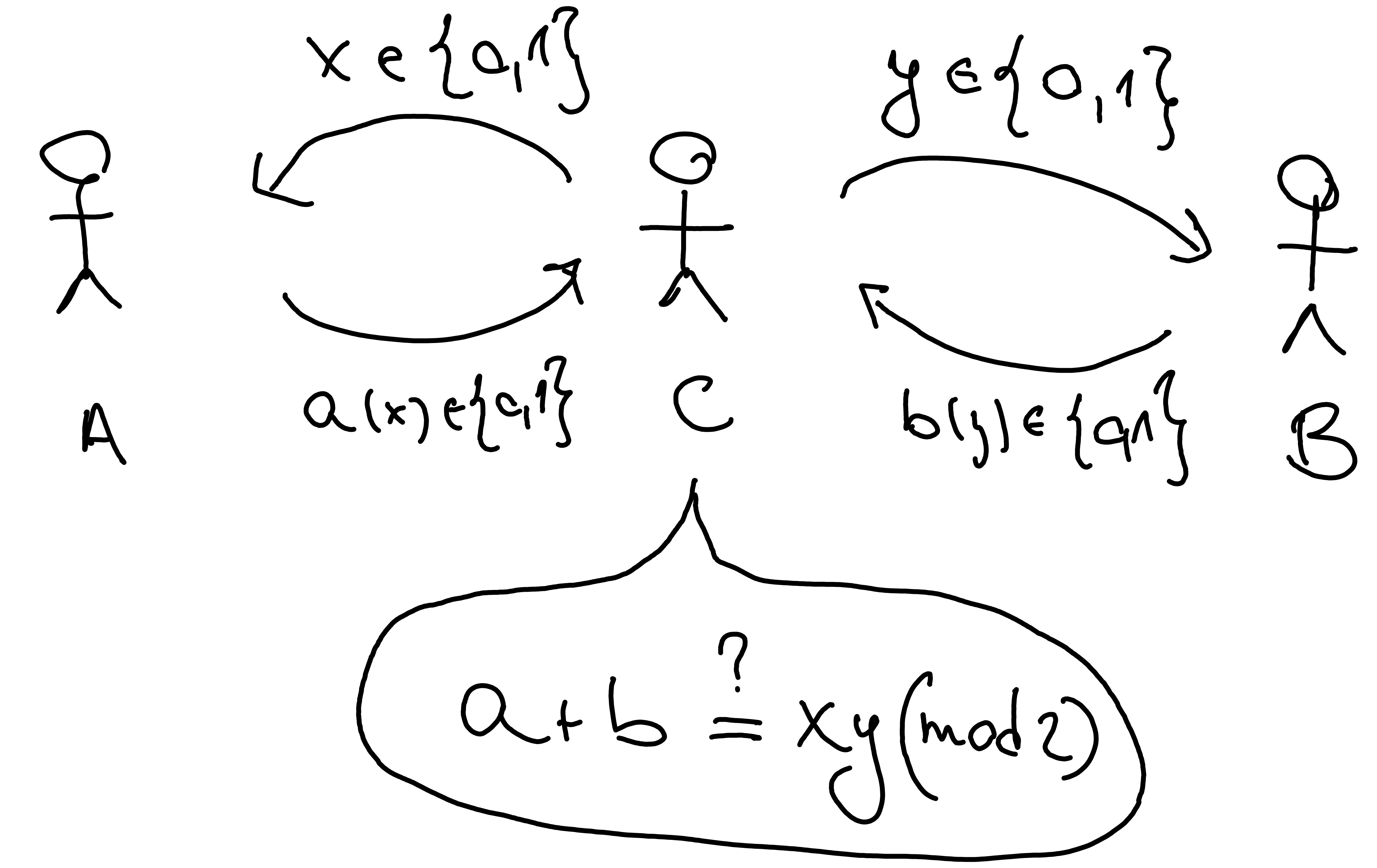
Goal: to understand entanglement: how different is it from classical physics?

Bell inequalities are a family of experiments aiming to clarify this.

We will learn the CHSH inequality named after Clauser, Horne, Shimony, Holt.

Best explained as a "game":

It involves 3 people : A, B, C.



- C sends 1 bit, x, to A and 1 bit, y, to B, randomly. The bits are independent and uniformly distributed.
- A & B look at their own bit, send back an answer to C
- C checks whether $a+b = xy \pmod 2$.
  If yes: A & B win 1€, no: A&B lose 1€.

<u>Goal</u>: strategy of A & B s.t. they win the most.

They can also do probabilistic

strategies! What matters: proba of answer a and b given that they received x & y: $p(ab|xy)$.

The exp. value of the game is then:

$$E = \sum_{abxy} (-1)^{a+b-xy} \, p(ab|xy).$$

For example, if A's decision depends only on x, B's decision on y, then:

$$p(ab|xy) = p_A(a|x) \cdot p_B(b|y).$$

But they can pre-share a correlated random variable, and base their decision on the value of that as well:

$$p(ab|xy) = \sum_{\lambda} q(\lambda) \cdot p_A(a|x,\lambda) \, p_B(b|y,\lambda).$$

This is a Local Hidden Variable, LHV, model. Classically we allow these kind of strategies.

We can upper bound E:

$$E = \sum_{a,b,x,y \in \{0,1\}} (-1)^{a+b-xy} \, p(a,b|x,y) =$$

$$= \sum_{abxy} (-1)^{a+b-xy} \sum_{\lambda} p_A(a|x,\lambda)\, p_B(b|y,\lambda)\, q(\lambda)$$

$$= \sum_{xy} (-1)^{xy} \sum_{\lambda} q_\lambda \underbrace{\sum_a (-1)^a p(a|x,\lambda)}_{A_x(\lambda) \in [-1,1]} \cdot \underbrace{\sum_b (-1)^b p(b|y,\lambda)}_{B_y(\lambda) \in [-1,1]}$$

$$= \sum_\lambda q_\lambda \sum_{xy} (-1)^{xy} A(x,\lambda)\, B(y,\lambda) =$$

$$= \sum_\lambda q_\lambda \Big(A_0(\lambda) B_0(\lambda) + A_0(\lambda) B_1(\lambda) + A_1(\lambda) B_0(\lambda) - A_1(\lambda) B_1(\lambda) \Big) =$$

$$= \sum_\lambda q_\lambda \Big[\big(A_0(\lambda) + A_1(\lambda)\big) B_0(\lambda) + \big(A_0(\lambda) - A_1(\lambda)\big) B_1(\lambda) \Big] \leq$$

$$\leq \sum_\lambda q_\lambda \Big[|A_0(\lambda) + A_1(\lambda)| \cdot |B_0(\lambda)| + |A_0(\lambda) - A_1(\lambda)| \cdot |B_1(\lambda)| \Big] \leq$$

$$\leq \sum_\lambda q_\lambda \Big[|A_0(\lambda) + A_1(\lambda)| + |A_0(\lambda) - A_1(\lambda)| \Big] \leq$$

$$\leq \sum_{\lambda} q_{\lambda} \cdot 2\max\left\{ |A_0(\lambda)|, |A_1(\lambda)| \right\} \leq 2.$$

So given a strategy allowed by classical physics, $E \leq 2$.

We can do better if we allow for A and B to share an entangled quantum state instead of a (classical) random variable.

<u>Quantum strategy</u>

- A & B share an entangled state
- They are allowed to measure their own part of the state. They will each have 2 measurements; they perform the measurement depending on the bit they see and send back the outcome of the measurement.

They will have projective measurements.

- If A sees $x=0$, she performs the measurement

$$\{A_0^0, A_0^1\}$$

- If A sees $x=1$, she performs the measurement

$$\{A_1^0, A_1^1\}$$

Same for B, he measures $\{B_0^0, B_0^1\}$ or $\{B_1^0, B_1^1\}$.

These are proj. meas.,

* $(A_i^j)^2 = (A_i^j)^+ = A_i^j$
* $\sum_j A_i^j = \mathbb{1}$.

Same for B.

Then the probability that they answer a&b given they have seen bits x&y is

$$p(ab|xy) = \mathrm{tr}\{(A_x^a \otimes B_y^b)\rho\}.$$

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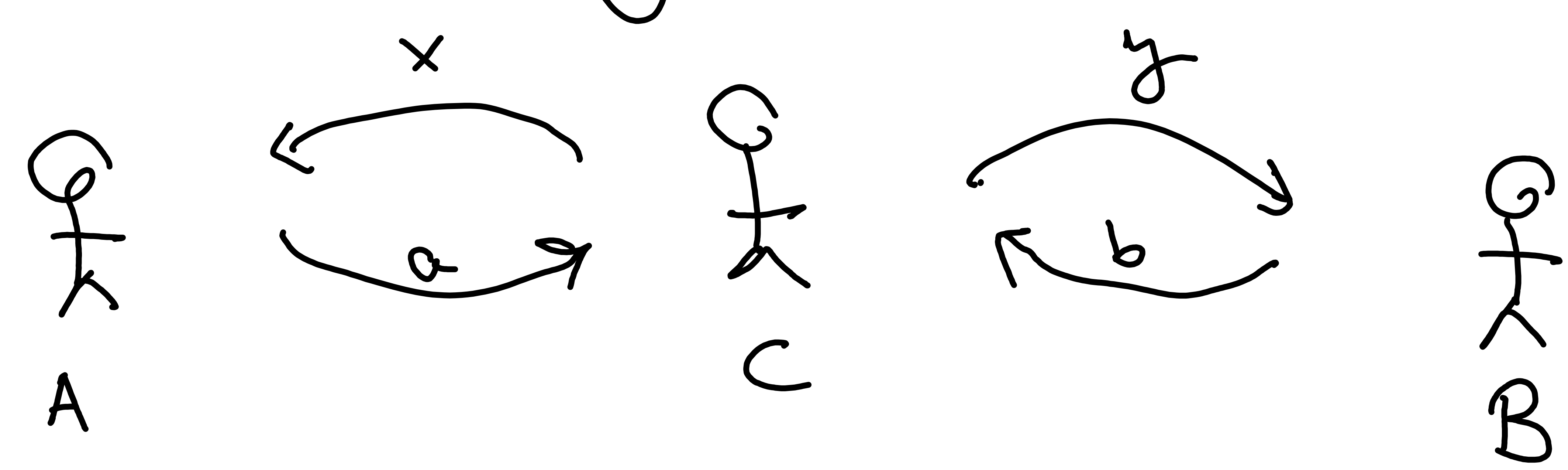
CHSH inequality



C sends to A & B bits x and y, respectively: uniform, independent (ie. each x, y pair w $1/4$ proba)

They can each look at their bits (x & y), and depending on what they see (no communication!) they send a one-bit reply: A sends a, B sends b.

<u>Note</u>: Bit: $x, y, a, b \in \mathbb{Z}_2$ ($\{0,1\}$ w/ binary addition and multiplication; $+ = \text{XOR}$, $\cdot = \text{AND}$).

Value of the game:

if $a + b = xy$, then A & B win 1 €

if $a + b \neq xy$, then A & B lose 1 €.

<u>Probabilistic strategies</u>: to characterize a strategy, we need to give $p(a, b | x, y)$:

the probability of answering a and b given A sees x and B sees y.

<u>Remark</u>: not all $p(a,b|x,y)$ are valid strategies: A and B are not allowed to communicate — B can't know X and A can't know y.

The goal of A & B is to win the most money:

$$E = \sum_{abxy} \underbrace{(-1)^{a+b-xy}}_{\substack{-1 \text{ if } a+b \neq xy, \\ +1 \text{ if } a+b = xy, \\ \text{i.e. how their money changes}}} \underbrace{p(ab|xy)\frac{1}{4}}_{\substack{\text{the proba. with} \\ \text{which } abxy \\ \text{happens,}}} = \max.$$

<u>Allowed classical strategies</u>:

(1) Before the game A & B gathers, they flip an uneven, many-sided die. They receive outcome $\lambda \in \{0,1,..,n\}$ for some $n \in \mathbb{N}$ with proba $q(\lambda)$.

(2) When they see the bits:

A sees x and knows λ: she flips an uneven coin, the coin chosen depending on x and λ and sends back the outcome.

B similarly.

Then, if A's coins have distr. $p_A(a|x,\lambda)$ and B's coins have distr. $p_B(y,\lambda)$, then

$$p(ab|x,y) = \sum_{\lambda} q(\lambda) \cdot p_A(a|x,\lambda)\, p_B(b|y,\lambda).$$

We have seen that $4 \cdot E \leq 2$.

<u>Allowed quantum strategies</u>

(1) Before the game A & B gather, they prepare a shared quantum state ρ_{AB}: $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, A physically holds one part of the state, B the other.

(2) A & B each fix 2 sets of <u>projective</u> measurement w/ outcome 0/1

A: $\{A_0^0, A_0^1\}$ and $\{A_1^0, A_1^1\}$

B: $\{B_0^0, B_0^1\}$ and $\{B_1^0, B_1^1\}$.

When A sees x, measures $\{A_x^0, A_x^1\}$ on ρ_{AB} and sends back the outcome.

When B sees y, measures $\{B_y^0, B_y^1\}$ on ρ_{AB} and sends back the outcome.

This strategy has the following cond. proba:

$$p(ab|xy) = \mathrm{tr}\{(A_x^a \otimes B_y^b)\rho_{AB}\},$$ as if A sees x and B sees y, together they do the measurement:

$$\{A_x^0 \otimes B_y^0, A_x^0 \otimes B_y^1, A_x^1 \otimes B_y^0, A_x^1 \otimes B_y^1\}.$$

<u>H(W)</u>: verify that this is a projective measurement.

Analysis of the strategy:

$$4E = \sum_{xyab} (-1)^{a+b-xy} p(ab|xy) =$$

$$= \sum_{xyab} (-1)^{a+b-xy} \operatorname{tr}\{(A_x^a \otimes B_y^b)\rho_{AB}\}$$

$$= \sum_{xy} (-1)^{xy} \sum_{ab} (-1)^a (-1)^b \operatorname{tr}\{(A_x^a \otimes B_y^b)\rho_{AB}\}$$

$\uparrow$

$(-1)^{-xy} = (-1)^{xy}$

$$= \sum_{xy} (-1)^{xy} \operatorname{tr}\Big\{ \underbrace{\Big(\sum_a (-1)^a A_x^a\Big)}_{:= A_x} \otimes \underbrace{\Big(\sum_b (-1)^b B_y^b\Big)}_{:= B_y} \rho_{AB} \Big\}$$

$$= \sum_{xy} (-1)^{xy} \operatorname{tr}\{(A_x \otimes B_y)\rho_{AB}\}.$$

Note that

$$A_x^2 = (A_x^0 - A_x^1)^2 = (A_x^0)^2 + (A_x^1)^2 - A_x^0 A_x^1 +$$
$$- A_x^1 A_x^0 = A_x^0 + A_x^1 = \mathbb{1}.$$

Note: $A_x^0 + A_x^1 = \mathbb{1}$, and they are proj $\Rightarrow$ $A_x^0 A_x^1 = 0$.

Similarly, $B_y^2 = \mathbb{1}$.

Note that $E(\rho)$ is convex $\Rightarrow$ extremal value is attained at an extremal point of $S(\mathcal{H}_A \otimes \mathcal{H}_B)$. So the best is to share a pure state!

$$|4E(\rho)| \leq \sup_{\substack{|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \\ \||\psi\rangle\| = 1}} \left| \sum_{xy} (-1)^{xy} \langle \psi | A_x \otimes B_y |\psi\rangle \right|,$$

where $A_x^2 = \mathbb{1}$ and $B_y^2 = \mathbb{1}$.

We thus obtain:

$$|4E(\rho)| \leq |\langle \psi | A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 |\psi\rangle|$$

Using Cauchy-Schwartz:

$$|\langle \psi | O | \psi\rangle|^2 \leq \underbrace{\langle \psi | \psi \rangle}_{1} \cdot \langle \psi | O^\dagger O | \psi \rangle$$

Let $O = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 =$

$$= A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1).$$

We obtain

$$OO^{\dagger} = O^2 = A_0^2 \otimes (B_0+B_1)^2 + A_1^2 \otimes (B_0 - B_1)^2$$

$$+ A_0 A_1 \otimes (B_0+B_1)(B_0-B_1) +$$

$$+ A_1 A_0 \otimes (B_0 - B_1)(B_0 + B_1)$$

$$= \mathbb{1} \otimes \Big[\underbrace{(B_0+B_1)^2 + (B_0-B_1)^2}_{2B_0^2 + 2B_1^2 = 4\mathbb{1}} \Big]$$

$$+ A_0 A_1 \otimes \big(\underbrace{B_0^2 - B_1^2}_{0} + B_1 B_0 - B_0 B_1 \big)$$

$$+ A_1 A_0 \otimes \big(\underbrace{B_0^2 - B_1^2}_{0} + B_0 B_1 - B_1 B_0 \big)$$

$$= 4\mathbb{1} + A_0 A_1 \otimes B_1 B_0 - A_0 A_1 \otimes B_0 B_1$$

$$+ A_1 A_0 \otimes B_0 B_1 - A_1 A_0 \otimes B_1 B_0 .$$

Here $\|A_0 A_1\| \leq \|A_0\| \|A_1\| = 1$

$\|B_1 B_0\| \leq \|B_1\| \|B_0\| = 1,$

Thus $\|A_0 A_1 \otimes B_1 B_0\| = \|A_0 A_1\| \cdot \|B_1 B_0\| \leq \mathbb{1}.$

Similar for all other terms.

Therefore

$$[4E(\rho)]^2 \leq \sup_{\substack{|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \\ \|\psi\| = 1}} \langle \psi | 4\mathbb{1} + A_0 A_1 \otimes B_1 B_0 - A_0 A_1 \otimes B_0 B_1$$

$$+ A_1 A_0 \otimes B_0 B_1 - A_1 A_0 \otimes B_1 B_0 | \psi \rangle \leq$$

$$\leq 4 + \sup_{\|\psi\|=1} \langle\psi| A_0A_1 \otimes B_1B_0|\psi\rangle + \sup_{\|\psi\|=1} \langle\psi|A_0A_1\otimes B_0B_1|\psi\rangle$$

$$\sup_{\|\psi\|=1} \langle\psi| A_1A_0 \otimes B_1B_0|\psi\rangle + \sup_{\|\psi\|=1} \langle\psi|A_1A_0\otimes B_0B_1|\psi\rangle$$

$$\leq 8.$$

Therefore $|4E(\rho)| \leq 2\sqrt{2}$

We can actually reach equality:

- $A_0 = X,\ A_1 = Z,\quad B_0 = \frac{X+Z}{\sqrt{2}},\ B_1 = \frac{X-Z}{\sqrt{2}}$
- $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, i.e., A & B share this state at the beginning.

To check that this works, we note

$$4E(\rho) = \langle\psi|\ \underbrace{A_0\otimes(B_0+B_1)}_{X\otimes X\cdot\sqrt{2}} + \underbrace{A_1\otimes(B_0-B_1)}_{Z\otimes Z\cdot\sqrt{2}}\ |\psi\rangle$$

Here, $X\otimes X + Z\otimes Z = \begin{pmatrix} & & & 1\\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1\end{pmatrix} =$

$$= \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}$$ and $|\psi\rangle$ is an eig. state

w/ eig. value 2, so $4E(\rho) = 2\sqrt{2}$.

Note that in order for A & B to do the best they could, they need to share an entangled state.

Actually, separable states give back the classical strategy: if $\rho_{AB} = \sum_i q_i \nu_i \otimes \eta_i$ with $\nu_i \in S(\mathcal{H}_A)$, $\eta_i \in S(\mathcal{H}_B)$ and $q_i \geq 0$, then

$$p(ab|xy) = \mathrm{tr}\left\{ \left(A_x^a \otimes B_y^b\right)\left(\sum_i q_i \nu_i \otimes \eta_i\right)\right\} =$$

$$= \sum_i q_i \,\mathrm{tr}\left\{ \left(A_x^a \otimes B_y^b\right)\left(\nu_i \otimes \eta_i\right)\right\} =$$

$$= \sum_i q_i \,\mathrm{tr}\left\{ A_x^a \nu_i \otimes B_y^b \eta_i \right\} =$$

$$= \sum_i q_i \underbrace{\mathrm{tr}\left\{ A_x^a \nu_i \right\}}_{p_A(a|x,i)} \cdot \underbrace{\mathrm{tr}\left\{ B_y^b \eta_i \right\}}_{p_B(b|y,i)}$$

$$= \sum_i q_i \, p_A(a|x,i)\, p_B(b|y,i).$$

This means that A & B share a separable state ρ at the beginning, then $4E(\rho) \leq 2$, they don't gain any advantage!

Remark:

Actually, any classical strategy can be reformulated as a quantum strategy, where A & B share an entangled state.

Main idea: if $\rho = \sum_i p_i |i\rangle\langle i|$ and we measure in comp. basis, then the outcome distr. is given by p_i.

Homework: reformulate classical strategy as a certain quantum strategy.

We can thus view entanglement as a resource: we can do things with it that are not possible w/o entanglement.

Next we learn some applications of entanglement: