# Entanglement conversion and classification

We have seen: entanglement gave us power to do more than what is possible classically.
I have always used max. ent. state. Are other ent. states as good? NO!
We will try to understand "how well" a state is entangled, i.e. try to quantify entanglement.
Careful! The entanglement quantification is application dependent, there is no single best answer to this problem.

We restrict ourselves to the bipartite scenario, as multi-partite entanglement is difficult.
Main tool we need: Schmidt decomposition.

Motivation: We want an ordering of ent. states (which one is better than the other). This ordering should not change under local unitaries: the entanglement of $(U \otimes V)|\psi\rangle$ is the same as that of $|\psi\rangle$, as whatever we can do w/ one state we can do w/ the other.

In particular, the max. ent. state $|\Omega\rangle$ is not unique: it is basis dependent and the basis on the 2 sides can be different. So anything of the form $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |v_i\rangle \otimes |w_i\rangle$ is max. ent. if $\{|v_i\rangle\}_{i=1}^{d}$ and $\{|w_i\rangle\}_{i=1}^{d}$ form an ONB.

In other words, $(U \otimes V)|\Omega\rangle$ is max. ent. if $|\Omega\rangle$ is max ent. and $U$ & $V$ are unitaries.

We want to get a property of $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ that is invar. under $|\psi\rangle \mapsto (U \otimes V)|\psi\rangle$.

<u>Definition</u> (<u>Schmidt decomposition</u>):

Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$. Then $\{(\lambda_i, \varphi_i, \chi_i)\}_{i=1}^n \subseteq \mathbb{C} \times \mathcal{H} \times \mathcal{H}$ is called a Schmidt decomposition if

(1) $\{\varphi_i\}_{i=1}^n \subseteq \mathcal{H}$ are orthonormal

(2) $\{\chi_i\}_{i=1}^n \subseteq \mathcal{K}$ are orthonormal

(3) $|\chi_i\rangle > 0 \;\; \forall i = 1, \dots, n$ (in part., $|\chi_i\rangle \in \mathbb{R}$)

(4) $|\psi\rangle = \sum_i \lambda_i |\varphi_i\rangle \otimes |\chi_i\rangle$.

The $\lambda_i$ are called Schmidt coefficients.

<u>Examples</u>:

(1) $|\Omega\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is a Schmidt decomposition

(2) $|\Omega\rangle = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$ is another Schmidt decomp. of the same state.

(3) $|\psi\rangle = \frac{1}{2}|0\rangle\otimes|0\rangle + \left(-\frac{1}{3}\right)|1\rangle\otimes|1\rangle$

is NOT a Schmidt decomposition.

Instead

$$|\psi\rangle = \frac{1}{2}\cdot|0\rangle\otimes|0\rangle + \frac{1}{3}\left(-|1\rangle\right)\otimes|1\rangle$$

is a Schmidt decomposition.

(4) $|\psi\rangle = \frac{1}{4}|+\rangle|0\rangle + \frac{1}{5}|-\rangle|1\rangle$ is a Schmidt decomp.

(5) $|\psi\rangle = \frac{1}{4}|+\rangle|0\rangle + \frac{1}{5}|-\rangle|0\rangle$ is NOT a Schmidt decomp. instead

$$|\psi\rangle = \text{norm}\cdot\frac{\left(\frac{1}{4}|+\rangle + \frac{1}{5}|-\rangle\right)}{\text{norm}}\otimes|0\rangle$$

is the Schmidt decomp.

Does a Schmidt decomposition exist?

Notice: if $|\psi\rangle = \sum_i \lambda_i |\varphi_i\rangle \otimes |\chi_i\rangle$ is a Schmidt decomposition, then

$$\mathrm{tr}_2 |\psi\rangle\langle\psi| = \mathrm{tr}_2\left(\sum_{ij} \lambda_i\lambda_j |\varphi_i\rangle\langle\varphi_j| \otimes |\chi_i\rangle\langle\chi_j|\right)$$

$$= \sum_{ij} \lambda_i\lambda_j |\varphi_i\rangle\langle\varphi_j| \cdot \langle\chi_j|\chi_i\rangle$$

$$= \sum_i \lambda_i^2 |\varphi_i\rangle\langle\varphi_i|.$$

This will help us to construct the Schmidt decomposition, proving its existence.

# Schmidt decomposition

$|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$, we write it as $|\psi\rangle = \sum_{i=1}^{n} \lambda_i |\varphi_i\rangle \otimes |\chi_i\rangle$,

- $\lambda_i > 0 \quad \forall i = 1..n$
- $\{|\varphi_i\rangle\}_{i=1}^{n} \subseteq \mathcal{H}$ are ortho-normal: $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$
- $\{|\chi_i\rangle\}_{i=1}^{n} \subseteq \mathcal{K}$ are ortho-normal. $\langle \chi_i | \chi_j \rangle = \delta_{ij}$

We have calculated, given existence of the Schmidt decomp:

$$\mathrm{tr}_K |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 |\varphi_i\rangle\langle\varphi_i|$$

This helps us to actually prove the existence:

**Thm** (Existence of Schmidt decomp)

Let $\mathcal{H}, \mathcal{K}$ be f.d. complex Hilbert spaces. Then $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$, $|\psi\rangle \neq 0$ admits a Schmidt decomposition.

**Proof**: Let $\rho = \mathrm{tr}_K |\psi\rangle\langle\psi|$. This $\rho$ is self-adjoint and in fact, positive semi-definite if $|\psi\rangle \neq 0$.

We can thus diagonalize it,

$$\rho = \sum_{i=1}^{n} \lambda_i^2 \, |\varphi_i\rangle\langle\varphi_i| , \quad w/ \ \lambda_i > 0.$$

$\{\varphi_i\}_{i=1}^{n}$ are orthonormal, we can extend it to a basis. Let us write thus

$$|\psi\rangle = \sum_{i=1}^{\dim(\mathcal{H})} |\varphi_i\rangle \otimes |\hat{\chi}_i\rangle$$

Notice now that

$$\rho = \sum_{ij} \langle \hat{\chi}_j | \hat{\chi}_i \rangle \cdot |\varphi_i\rangle\langle\varphi_j| ,$$

but $\rho$ is diagonal in this basis, so

$$\langle \hat{\chi}_j | \hat{\chi}_i \rangle = \lambda_i^2 \, \delta_{ij} \qquad (\lambda_i = 0 \text{ for } i > n)$$

We thus obtain that

(1) $n \leq \min(\dim(\mathcal{H}), \dim(\mathcal{K}))$

(2) for $i \leq n$ we can write $|\hat{\chi}_i\rangle = \lambda_i |\chi_i\rangle$, with $|\chi_i\rangle$ orthonormal,

and thus

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i \, |\varphi_i\rangle \otimes |\chi_i\rangle \text{ is a Schmidt}$$

decomposition. $\square$

Remark: $\mathrm{tr}_1 |\psi\rangle\langle\psi|$ and $\mathrm{tr}_2 |\psi\rangle\langle\psi|$ has the same spectrum (apart from 0)

Remark: if $\mathcal{K} = \mathcal{H}^*$, $\mathcal{H} \otimes \mathcal{H}^* \cong \mathcal{B}(\mathcal{H})$, and we obtain for a matrix $M \in \mathcal{B}(\mathcal{H})$

$$M = \sum_i \lambda_i |u_i\rangle\langle v_i| =$$

$$= \left(\sum_i |u_i\rangle\langle i|\right) \cdot \left(\sum_i \lambda_i |i\rangle\langle i|\right) \cdot \left(\sum_i |i\rangle\langle v_i|\right) = UDV,$$

the Schmidt decomp. is called SVD, Singular Value Decomposition.

$$\|M\| = \sup_{\||\psi\rangle\| = 1} \langle\psi|M^+M|\psi\rangle = \text{largest sing. value of } M.$$

Eg: $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow$ eig. values : 0

$M^+M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ eig. values = sing. values of $M$ : $\{1, 0\}$.

$\|M\| = 1.$

Remark: from uniqueness of sp. decomp. we see that if $\lambda_i$ are distinct, then the Schmidt decomp. is unique.

$\boxed{HW}$ What happens if $\lambda_i$ are degenerate?

Remark: $|\psi\rangle$ and $(U\otimes V)|\psi\rangle$ has same Schmidt values.

Back to entanglement:

We want to characterize when $|\psi\rangle$ is more entangled than $|\phi\rangle$. In all applications (CHSH, teleportation) we are allowed to do local unitaries:

$|\psi\rangle$ and $(U\otimes V)|\psi\rangle$ is equally entangled.

So far entanglement classification (in the pure, bipartite case) only the Schmidt coefficients matter!

What do we expect from ent. classification?

Resource-theoretical P.O.V: if

$$|\psi\rangle \xrightarrow{\text{"classical operations"}} |\phi\rangle,$$ then

$|\phi\rangle$ is less entangled than $|\psi\rangle$.

Which operations are "classical"?

Application dependent. For communication scenarios,

- Local operations (unitaries, measurements)
- Classical communication (of meas. outcomes)

$\Big\}$ LOCC

are "classical". If $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$, then $|\psi\rangle$ is better (not worse).

<u>Example LOCC</u> : A & B share $|\psi\rangle$.

- A measures $\{M_i\}_{i=1}^n$, receives outcome $i$. The post-measurement state is then

$$\frac{1}{\sqrt{p_i}}(M_i \otimes \mathbb{1})|\psi\rangle$$

- Now A sends "$i$" to B, B performs a unitary $U_i$ depending on $i$ (here we assume that B also knows $|\psi\rangle$ and $M_i$ - but this info is available beforehand).

The outcome is

$$|\psi\rangle \longmapsto \frac{1}{\sqrt{p_i}}(M_i \otimes U_i)|\psi\rangle.$$

We say that $|\psi\rangle \longrightarrow |\phi\rangle$ under this protocol if $\frac{1}{\sqrt{p_i}}(\Pi_i \otimes U_i)|\psi\rangle = |\phi\rangle \;\forall i$.

In general, an LOCC protocol consists of multiple rounds of such operations.

E.g. first A measures, B applies unitary, then B measures, A applies unitary, e.t.c.

We say that $|\psi\rangle \longrightarrow |\phi\rangle$ under this protocol if for <u>all</u> outcomes we obtain $|\phi\rangle$ at the end.

One can prove, however, that in this scenario one round of measurement-unitary is enough.

That is,

<u>Thm</u>: Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ iff. $\exists \; \{\Pi_i\}_{i=1}^{n} \subseteq B(\mathcal{H}_A)$ POVB, $\{U_i\}_{i=1}^{n} \subseteq B(\mathcal{H})$ unitaries and $\{p_i\}_{i=1}^{n} \subseteq \mathbb{R}_{+,0}$ s.t.

$$\sqrt{p_i}\,|\phi\rangle = (\Pi_i \otimes U_i)\,|\psi\rangle \quad \forall i = 1..n.$$

<u>Proof</u>: See exercise sheet.

Remark: for $n=1$, we obtain $|\psi\rangle \longrightarrow (U\otimes V)|\psi\rangle$ for $U, V$ unitaries. Concatenation of LOCCs is LOCC, so whether $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ is possible, only depends on the Schmidt coefficients of $|\psi\rangle$ and $|\phi\rangle$. What is this dependence? Let us first show a necessary criterion. For that, note

Lemma (Ky-Fan) Let $\mathcal{H}$ be a Hilbert space,

$$P_k(\mathcal{H}) := \{ P \in B(\mathcal{H}) \mid P^\dagger = P = P^2,\ \mathrm{tr}(P) = k \}.$$

Let $\rho \in B(\mathcal{H})$ be a Hermitian operator,

$$\rho = \sum_{i=1}^{\dim(\mathcal{H})} \lambda_i |\phi_i\rangle\langle\phi_i|$$ its eig. decomp. w/

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then

$$\sup_{P\in P_k(\mathcal{H})} \mathrm{tr}\{\rho P\} = \sum_{i=1}^{k} \lambda_k .$$

Proof: $\boxed{\geq}$ Choose $P = \sum_{i=1}^{k} |\phi_i\rangle\langle\phi_i|$, then

$$\mathrm{tr}\{\rho P\} = \sum_{i=1}^{k} \lambda_k ,$$

$\boxed{\leq}$ $\mathrm{tr}\{\rho P\} \overset{k}{=} \sum_{i=1}^{n} \lambda_i \langle\phi_i|P|\phi_i\rangle = \sum_{i=1}^{n} \lambda_i w_i ,$

where $1 \geq w_i \geq 0$ and $\sum_{i=1}^{n} w_i = k$.

The max. is thus clearly at $w_1 = \dots = w_k = 1$, all else $0$.

$\square$.

Let us consider now $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$, i.e.,

$$(M_i \otimes U_i)|\psi\rangle = \sqrt{p_i}\,|\phi\rangle .$$

Trace out A in this equation:

$$p_i \, tr_A\{|\phi\rangle\langle\phi|\} = tr_A\{(M_i \otimes U_i)|\psi\rangle\langle\psi|(M_i^\dagger \otimes U_i^\dagger)\}$$

$$\underset{\uparrow}{=} \sum_j tr_A\{M_i A_j M_i^\dagger \otimes U_i B_j U_i^\dagger\} =$$

$|\psi\rangle\langle\psi| = \sum_j A_j \otimes B_j$

$$= \sum_j tr(M_i A_j M_i^\dagger)\, U_i B_j U_i^\dagger =$$

$$= \sum_j tr\{M_i^\dagger M_i A_j\}\, U_i B_j U_i^\dagger =$$

$$= U_i \rho_i U_i^\dagger, \text{ where}$$

$$\rho_i = \sum_j tr\{M_i^\dagger M_i A_j\} B_j = tr_A\{(M_i \otimes \mathbb{1})|\psi\rangle\langle\psi|(M_i^\dagger \otimes \mathbb{1})\} \geq 0$$

and $\sum_i \rho_i = \sum_{ij} tr(M_i^\dagger M_i A_j) B_j = \sum_j tr(A_j) B_j = tr_A(|\psi\rangle\langle\psi|)$.

So we obtain:

$$\mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_i \rho_i = \sum_i p_i \,\mathcal{U}_i\, \mathrm{tr}_A |\phi\rangle\langle\phi| \,\mathcal{U}_i^+ .$$

Let $\mathrm{tr}_A |\psi\rangle\langle\psi| = \rho$ and $\mathrm{tr}_A |\phi\rangle\langle\phi| = \eta$.

Then

$$\sup_{P\in P_k(\mathcal{H})} \mathrm{tr}\{\rho P\} = \sup_{P\in P_k(\mathcal{H})} \sum_i p_i \,\mathrm{tr}(P\mathcal{U}_i \eta \mathcal{U}_i^+)$$

$$\leq \sum_i p_i \sup_{P\in P_k(\mathcal{H})} \mathrm{tr}\{\mathcal{U}_i^+ P \mathcal{U}_i \eta) \leq$$

$$\leq \sum_i p_i \sup_{P\in P_k(\mathcal{H})} \mathrm{tr}\{P\eta\} = \sup_{P\in P_k(\mathcal{H})} \mathrm{tr}\{P\eta\} .$$

That is, if $|\psi\rangle = \sum_i \sqrt{p_i}\, |l_i\rangle \otimes |r_i\rangle$ and

$$|\phi\rangle = \sum_i \sqrt{q_i}\, |\hat{l}_i\rangle \otimes |\hat{r}_i\rangle ,$$

where we order the Schmidt coeffs in descending order, then, as $\rho$ has eig. values $p_i$ and $\eta$ has eig. values $q_i$,

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i$$

<u>Def</u> (majorization) Let $p, q \in \mathbb{R}^n$ be probability distributions. Let $p^\downarrow, q^\downarrow$ be the vector obtained from $p, q$ by listing their entries in descending order. Then we say $p \precsim q$, $p$ is majorized by $q$, if

$$\sum_{i=1}^{n} p_i^\downarrow \le \sum_{i=1}^{n} q_i^\downarrow .$$

We have thus seen:

<u>Prop</u>: Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Let the Schmidt decomp. be

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}\, |l_i\rangle \otimes |r_i\rangle$$

$$|\phi\rangle = \sum_{i=1}^{n} \sqrt{q_i}\, |\hat{l}_i\rangle \otimes |\hat{r}_i\rangle .$$

Then $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \Rightarrow p \precsim q$.

# Entanglement classification

- Pure, bipartite states
- Want to say: "$|\psi\rangle$ is more entangled than $|\varphi\rangle$", i.e. an ordering of the states.
- We have: $|\psi\rangle$ is entangled iff it is not a product state.
- Strategy: identify a large class of operations that we declare not to grow entanglement (this is LOCC), then we say $|\psi\rangle$ is more entangled than $|\phi\rangle$ if $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$.
- LOCC operations: operations that "do not grow entanglement" should bring product states to product states (they should not transform non-entangled state to entangled)

<u>Reminder</u>: operation (pure states): unitary or measurement.

## <u>Unitaries</u>:

Product unitaries bring prod state to prod,
$(U \otimes V)(|\varphi\rangle \otimes |\psi\rangle) = U|\varphi\rangle \otimes V|\psi\rangle$. So,

$$U \otimes V \in LOCC.$$

Global (not tensor prod.) unitaries $\notin$ LOCC.

Example: $CNOT = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X \notin LOCC$

as $CNOT(|+\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \notin SEP$,

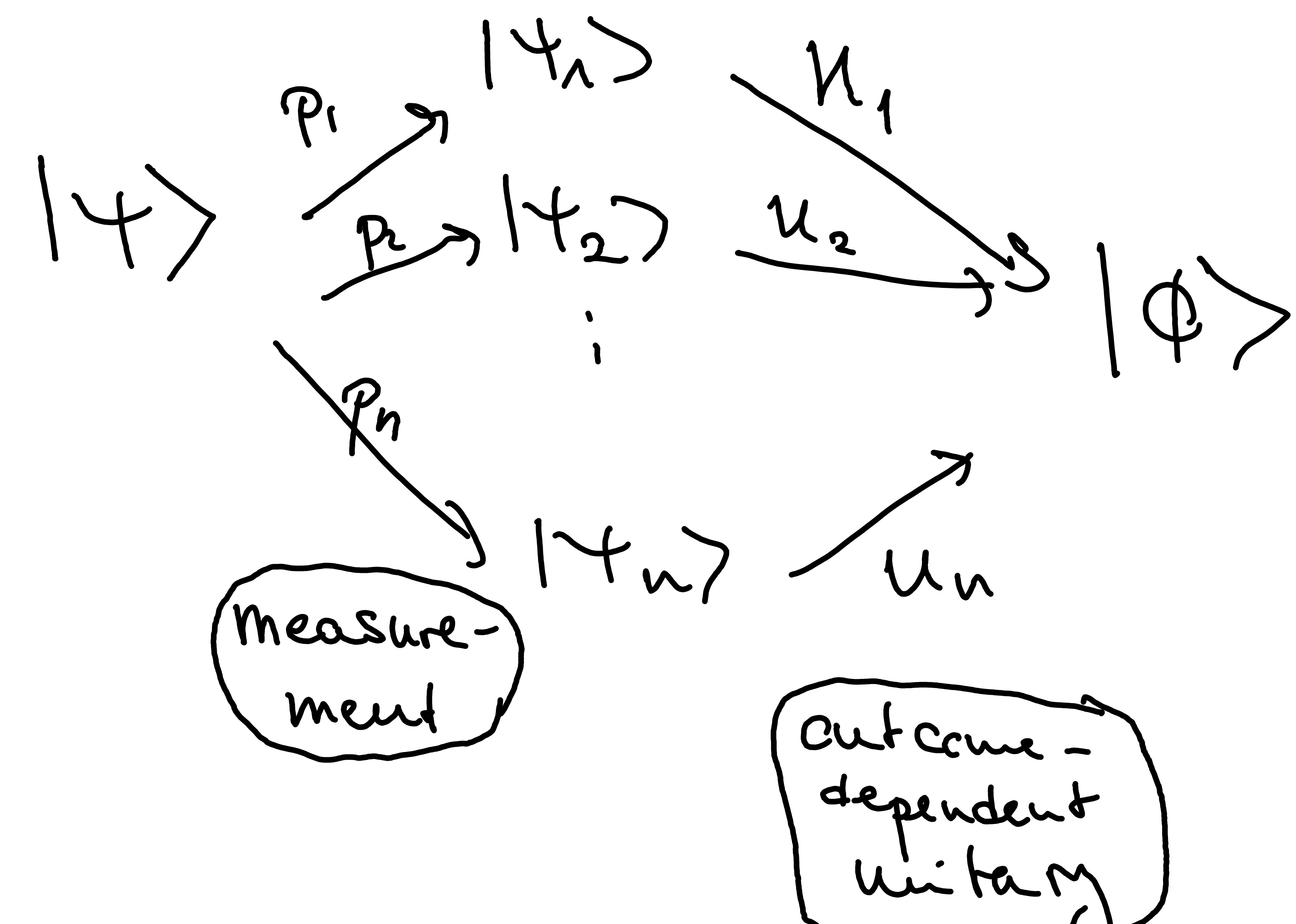
so CNOT brings a product state to an entangled state $\Rightarrow$ it is an <u>entangling</u> operation, can't be LOCC.

<u>General feeling</u>: Local operations (i.e., tensor product) does not grow entanglement, global operations might.

# Measurements

Problem: different outcomes transform the state differently, what does $|\psi\rangle \longrightarrow |\phi\rangle$ even mean?

Example: teleportation. There with extra operations that depend on the meas. outcome, we can reach always the same state!




Further problem: the correction might depend on the input state: if we define a protocol s.t. $|\psi\rangle \longrightarrow |\phi\rangle$, $|\psi'\rangle$ might transform to a different state for each output.

<u>Example</u>: start from $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Omega\rangle$,

A measures in comp. basis:

$$
\begin{tikzcd}
 & {|0\rangle \otimes |0\rangle} \arrow[dr, "\mathbb{1} \otimes \mathbb{1}"] & \\
{|\Omega\rangle} \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & & {|0\rangle \otimes |0\rangle} \\
 & {|1\rangle \otimes |1\rangle} \arrow[ur, "X \otimes X"'] &
\end{tikzcd}
$$

But if we start from $|\psi\rangle = |+\rangle \otimes |0\rangle$, then the same protocol leads to an ensemble:

$$
\begin{tikzcd}
 & {|0\rangle \otimes |0\rangle} \arrow[r, "\mathbb{1} \otimes \mathbb{1}"] & {|0\rangle \otimes |0\rangle} \\
{|\psi\rangle} \arrow[ur, "1/2"] \arrow[dr, "1/2"'] & & \\
 & {|1\rangle \otimes |0\rangle} \arrow[r, "X \otimes X"'] & {|0\rangle \otimes |1\rangle}
\end{tikzcd}
$$

So for different input we end up w/ non-trivial ensemble. Therefore the correct way to think about this operation is as a CP map; its output is a mixed state:

$$|\psi\rangle\langle\psi| \longmapsto |0\rangle\langle 0| \otimes \left(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\right)$$

The outcome is product, not entangled.

Important here: we do local measurement, then local unitaries (that depend on the outcome). One can show that such operations map separable states to separable.

Exercise:

- Remember the definition of entanglement and separability for mixed states
- Write the CP map that has the following action on a pure state
  - A measures $\{\Pi_i\}_{i=1}^n$
  - B applies unitary $\{U_i\}_{i=1}^n$
  - The outcome is the mixed state corresponding to the resulting ensemble.
- Show that such a map brings separable states to separable.

An LOCC operation is repeated such transformation:

$$
\begin{tikzcd}
 & \underline{\text{A measures}} & \underline{\text{Uniteries}} & \underline{\text{B measures}} \\
 & & & {|\psi_{1,1}\rangle} \\
 & {|\psi_1\rangle} \arrow[r] & {U_1 \otimes V_1 |\psi_1\rangle} \arrow[ur] \arrow[r, phantom, "\vdots"] \arrow[dr] & \vdots \\
{|\psi\rangle} \arrow[ur] \arrow[ddr] \arrow[r, phantom, "\vdots"] & \vdots & \vdots & {|\psi_{1,m_1}\rangle} \\
 & & & \cdots \\
 & {|\psi_n\rangle} \arrow[r] & {U_n \otimes V_n |\psi_n\rangle} \arrow[r] \arrow[dr, phantom, "\vdots"] \arrow[ddr] & {|\psi_{n,1}\rangle} \\
 & & & \vdots \\
 & & & {|\psi_{n,m_n}\rangle}
\end{tikzcd}
$$

After such a protocol, we end up in an ensemble in general. We say that

$|\psi\rangle \longrightarrow |\phi\rangle$ under this protocol, if the ensemble is trivial : 100% $|\phi\rangle$.

Easy to show (see Exercise for a single round): Such a protocol brings separable states to separable.

We thus declare: If $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$, then $|\psi\rangle$ is more entangled than $|\phi\rangle$.

We have seen (Exercise) that in the 2-partite case, $|\psi\rangle, |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ iff $\exists \{M_i\}_{i=1}^n \subseteq B(\mathcal{H}_A)$ measurement, $\{U_i\}_{i=1}^n \subseteq B(\mathcal{H}_B)$ unitaries and $\{p_i\}_{i=1}^n \subseteq \mathbb{R}$ proba distr. s.t.

$$\sqrt{p_i}\,|\phi\rangle = (M_i \otimes U_i)\,|\psi\rangle.$$

We have seen that this property depends only on the Schmidt coefficients of $|\phi\rangle$ and $|\psi\rangle$.

<u>Reminder</u>: Schmidt decomposition.

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{p_i}\; |l_i\rangle \otimes |r_i\rangle$$

- $\{|l_i\rangle\}_{i=1}^r$ ONS
- $\{|r_i\rangle\}_{i=1}^r$ ONS
- $p_i \geq 0$

**Def** (majorization) Let $p, q \in \mathbb{R}^n$ be probability distributions. Let $p^{\downarrow}, q^{\downarrow}$ be the vector obtained from $p, q$ by listing their entries in descending order. Then we say $p \preccurlyeq q$, $p$ is majorized by $q$, if

$$\sum_{i=1}^{n} p_i^{\downarrow} \leq \sum_{i=1}^{n} q_i^{\downarrow}.$$

**Prop**: Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Let the Schmidt decomp. be

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}\, |l_i\rangle \otimes |r_i\rangle$$

$$|\phi\rangle = \sum_{i=1}^{n} \sqrt{q_i}\, |\hat{l}_i\rangle \otimes |\hat{r}_i\rangle.$$

Then $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle \Rightarrow p \preccurlyeq q$.

**Example**: $|\phi\rangle = |\Omega\rangle$, max. ent. state in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

$$p \preccurlyeq \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \Rightarrow p_1 \leq \tfrac{1}{2}, \text{ but } p_1 \geq p_2$$

and $p_1 + p_2 = 1 \Rightarrow p_1 \geq \frac{1}{2}$ too, so $p_1 = \frac{1}{2}$.

Same way, if $q = (1,0)$, then

$q \succcurlyeq p$ for all proba distr.

In general, $(1,0,\dots,0) \succcurlyeq p$ for all proba distr. as $1 \geq p_1^\downarrow + \dots + p_k^\downarrow \quad \forall k = 1 \dots n$.

On the other end, $\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right) \preccurlyeq p$ for all proba distr. as

$$p_1^\downarrow + \dots + p_k^\downarrow \geq \frac{1}{\binom{n}{k}} \underbrace{\sum_{i_1 \geq \dots \geq i_k} p_{i_1}^\downarrow + \dots + p_{i_k}^\downarrow}_{\substack{\text{all possible choices} \\ \text{of indices, there is } \binom{n}{k} \\ \text{such choice.}}} =$$

$$= \frac{1}{\binom{n}{k}} \cdot \underbrace{\sum_i p_i^\downarrow \binom{n-1}{k-1}}_{\substack{\text{in the sum before,} \\ p_i^\downarrow \text{ appeas } \binom{n-1}{k-1} \text{ times}}} = \frac{k}{n}.$$

Note that majorization is a <u>partial</u> order. For example, can't compare

$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ and $\left(\frac{7}{16}, \frac{7}{16}, \frac{1}{8}\right)$:

$1/2 > 7/16$ but $\frac{1}{2} + \frac{1}{4} = \frac{12}{16} < \frac{14}{16}$.

We have thus

(1) $|\psi\rangle \otimes |\chi\rangle \xrightarrow{LOCC} |\phi\rangle$ only if $|\phi\rangle$ is product

(2) $|\psi\rangle \xrightarrow{LOCC} |\Omega\rangle$ max ent. only if $|\psi\rangle = (U \otimes V)|\Omega\rangle$.

Let us now try to understand a sufficient criterion for $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$, i.e., construct explicit $\{M_i\}_{i=1}^n$ POVM and $\{U_i\}_{i=1}^n$ unitaries s.t.

$$(M_i \otimes U_i)|\psi\rangle = \sqrt{p_i}\,|\phi\rangle.$$

Example: $|\psi\rangle = |\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\phi\rangle = \sqrt{\frac{2}{3}}\,|00\rangle + \sqrt{\frac{1}{3}}\,|11\rangle.$

Let the POVM $\{M_0, M_1\}$ be:

$$M_0 = \begin{pmatrix} \sqrt{2/3} & \\ & \sqrt{1/3} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \sqrt{1/3} & \\ & \sqrt{2/3} \end{pmatrix}.$$

Then the post-meas. states are

$|\psi_0\rangle = \sqrt{\frac{2}{3}}\,|00\rangle + \sqrt{\frac{1}{3}}\,|11\rangle$

$|\psi_1\rangle = \sqrt{\frac{1}{3}}\,|00\rangle + \sqrt{\frac{2}{3}}\,|11\rangle.$

If outcome $= 0 \Rightarrow$ nothing to do.

If outcome $= 1 \Rightarrow$ Apply $X \otimes X$, local op.

<u>Essence</u>: $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{1+X}{2} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

Works for any convex combinations of permutations! That is, let

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |l_i\rangle \otimes |r_i\rangle,$$

$$|\phi\rangle = \sum_i \sqrt{\mu_i}\, |\hat{l}_i\rangle \otimes |\hat{r}_i\rangle$$

be the Schmidt decompositions of $|\psi\rangle$ and $|\phi\rangle$.

Assume $\lambda = \sum_\pi p_\pi \pi \mu$, where $p$ is a proba distr. and $\pi$ are permutations.

Then $|\psi\rangle \xrightarrow{\text{LOCC}} |\phi\rangle$.

<u>Proof</u>: Let us define a POVM as

$$M_\pi = \sum_i \left( p_\pi \frac{\mu_{\pi(i)}}{\lambda_i} \right)^{1/2} |\hat{l}_{\pi(i)}\rangle\langle l_i| .$$

This is a POVM as

$$\sum_\pi M_\pi^\dagger M_\pi = \sum_{ik} \underbrace{\sum_\pi p_\pi \frac{\mu_{\pi(i)}}{\lambda_i}}_{1} |l_i\rangle \underbrace{\langle \hat{l}_{\pi(i)} | \hat{l}_{\pi(k)} \rangle}_{\delta_{ik}} \langle l_k| = \mathbb{1}$$

Also,

$$(M_\pi \otimes \mathbb{1})|\psi\rangle = \sum_i \sqrt{p_\pi \mu_{\pi\cdot(i)}}\, |\hat{\ell}_{\pi\cdot(i)}\rangle \otimes |r_i\rangle$$

Therefore, if we set

$$V_\pi = \sum_i |r_{\pi(i)}\rangle\langle r_i|,$$

then $V_\pi^\dagger V_\pi = \mathbb{1}$ and

$$(M_\pi \otimes V_\pi)|\psi\rangle = \sqrt{p_\pi} \sum_i \sqrt{\mu_{\pi(i)}}\, |\hat{\ell}_{\pi(i)}\rangle \otimes |\hat{r}_{\pi(i)}\rangle$$

$$= \sqrt{p_\pi} \sum_i \sqrt{\mu_i}\, |\hat{\ell}_i\rangle \otimes |\hat{r}_i\rangle$$

$$= \sqrt{p_\pi}\, |\phi\rangle. \qquad \square$$

<u>Lemma</u>: Let $p, q$ be probability distributions. If $p \preceq q$, then there are $\{\Pi_i\}_{i=1}^m$ permutations, $r \in \mathbb{R}^m$ proba distr. s.t.

$$p = \sum_i r_i \Pi_i q.$$

We have thus seen:

<u>Prop</u>: Let $|\phi\rangle, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ w/ Schmidt decomp.

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}\, |l_i\rangle \otimes |r_i\rangle$$

$$|\phi\rangle = \sum_{i=1}^{n} \sqrt{q_i}\, |\hat{l}_i\rangle \otimes |\hat{r}_i\rangle.$$

Then, if $p \preceq q$, then $|\psi\rangle \xrightarrow{\text{Locc}} |\phi\rangle$.

The two props together read as:

<u>Thm</u>: Let $|\phi\rangle, |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ w/ Schmidt decomp.

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}\, |l_i\rangle \otimes |r_i\rangle$$

$$|\phi\rangle = \sum_{i=1}^{n} \sqrt{q_i}\, |\hat{l}_i\rangle \otimes |\hat{r}_i\rangle.$$

Then $|\psi\rangle \xrightarrow{\text{Locc}} |\phi\rangle$ iff $p \preceq q$.

So we say $|\psi\rangle$ is more entangled than $|\phi\rangle$ iff the Schmidt values (square) of $|\phi\rangle$ majorize that of $|\psi\rangle$.