Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2024/25

— Exercise Sheet #1 —

Problem 1: Pauli matrices

Recall the Pauli matrices from the lecture, which in the computational basis $\{|0\rangle, |1\rangle\}$ are of the form

$$X = \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad Y = \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

1. Show that the Pauli matrices are all Hermitian, unitary, square to the identity, and different Pauli matrices anticommute, i.e., $\sigma_i\sigma_j = -\sigma_j\sigma_i$ if $i \neq j$.
2. Check the relation $\sigma_\alpha\sigma_\beta = \sum_\gamma i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma + \delta_{\alpha\beta}I$ $(\alpha, \beta, \gamma = 1, 2, 3)$, where
   - $\delta_{\alpha\beta}$ is the Kroenecker delta, i.e., $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise,
   - and $\varepsilon_{\alpha\beta\gamma}$ is the fully antisymmetric tensor, i.e., $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$, and zero otherwise.
3. The *trace* $\mathrm{tr}[X]$ is defined as the sum of the diagonal elements of X, i.e., $\mathrm{tr}[X] := \sum_i X_{ii}$. Determine $\mathrm{tr}[I]$, $\mathrm{tr}[\sigma_\alpha]$, and $\mathrm{tr}[\sigma_\alpha\sigma_\beta]$.
4. Write each operator X, Y and Z using bra-ket notation with states from the computational basis.
5. Find the eigenvalues e_i and eigenvectors $|v_i\rangle$ $(i = 0, 1)$ for each Pauli matrix σ (expressed in the computational basis), and check that $\sigma = e_0|v_0\rangle\langle v_0| + e_1|v_1\rangle\langle v_1|$ holds.

Problem 2: Matrix spaces as Hilbert spaces

Let $\mathcal{V}_d$ be the space of all complex $d \times d$ matrices, and $\mathcal{W}_d \subset \mathcal{V}_d$ the space of all hermitian complex $d \times d$ matrices (i.e. for $M \in \mathcal{W}_d$, $M = M^\dagger$).

1. Show that $\mathcal{V}_d$ forms a vector space over $\mathbb{C}$, and $\mathcal{W}_d$ forms a vector space over $\mathbb{R}$, but not over $\mathbb{C}$: that is, in $\mathcal{V}_d$ you can take complex linear combinations, while in $\mathcal{W}_d$ only real linear combinations. In the following we will always consider $\mathcal{V}_d$ as a complex and $\mathcal{W}_d$ as a real vector space.
2. Show that the Pauli matrices together with the identity σ_0, $\Sigma := \{\sigma_i\}_{i=0}^3$, form a basis for both $\mathcal{V}_2$ (over $\mathbb{C}$) and $\mathcal{W}_2$ (over $\mathbb{R}$).
3. Show that
$$(A, B) = \mathrm{tr}[A^\dagger B]$$
defines a scalar product (the "Hilbert-Schmidt scalar product") both for $\mathcal{V}_d$ and for $\mathcal{W}_d$[1]. Here, $\mathrm{tr}[X]$ is the trace, i.e., the sum of the diagonal elements.
4. Show that the Pauli matrices Σ form an orthogonal basis with respect to the Hilbert-Schmidt scalar product.
5. Use the fact that for any scalar product $(\vec{v}, \vec{w})$ and a corresponding ONB $\vec{w}_i$, we can write
$$\vec{v} = \sum_i \vec{w}_i(\vec{w}_i, \vec{v}) ,$$
to express a general matrix in $M \in \mathcal{V}_2$ as
$$M = \sum m_i\sigma_i .$$
What is the form of the m_i? What special property do the m_i satisfy for $M \in \mathcal{W}_2$?

[1] A scalar product $(.,.)$ on a real vector space is bilinear, symmetric, i.e., $(A, B) = (B, A)$ and positive definite: $(A, A) \geq 0$ and $(A, A) = 0$ if and only if $A = 0$. Note that the only difference between real and complex scalar product is that in the complex case the scalar product is conjugate linear in the first variable, not linear.

6. Show that a hermitian orthonormal basis also exists for $\mathcal{V}_d$ and $\mathcal{W}_d$. (Ideally, explicitly construct such a basis.)

Problem 3: Eigenvectors

In the following $\mathcal{H}$ is a finite dimensional Hilbert space ($\mathbb{C}^d$ with the usual scalar product), and $\mathcal{B}(\mathcal{H})$ is the set of $d \times d$ matrices.

1. Let $A \in \mathcal{B}(\mathcal{H})$, and $A^\dagger$ its conjugate transpose. Show that $\langle w|Av\rangle = \langle A^\dagger w|v\rangle$ for any $|v\rangle, |w\rangle \in \mathcal{H}$.
2. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, i.e. $A = A^\dagger$. Using item 1, show that any eigenvalue of A is real.
3. Let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, i.e. $A = A^\dagger$. Using item 1, show that if $|v\rangle$ is an eigenvector with λ and $|w\rangle$ is an eigenvector with $\mu \neq \lambda$, then $\langle w|v\rangle = 0$.
4. Let $A \in \mathcal{B}(\mathcal{H})$, and let the set $\{\lambda_i\}_{i=1}^n$ be a subset of its eigenvalues. For each $i = 1, \dots, n$ let $|v_i\rangle$ be an eigenvector. Show that the set $\{|v_i\rangle\}_{i=1}^n$ is linearly independent. *Hint:* consider the polynomial[2] $\frac{(A-\lambda_2 I)(A-\lambda_3 I)\dots(A-\lambda_n I)}{(\lambda_1-\lambda_2)\dots(\lambda_n-\lambda_1)}$ acting on a linear combination of such vectors (here I is the identity matrix).
5. Let $A \in \mathcal{B}(\mathcal{H})$. Assume that it has d linearly independent eigenvectors, $|v_1\rangle, \dots |v_d\rangle$, with eigenvalues $\lambda_1, \dots, \lambda_d$. Let $X = \sum_i |v_i\rangle\langle i|$. Show that
   - X is invertible, and
   - if $|v_1\rangle, \dots |v_d\rangle$ are orthogonal, then X is unitary, and
   - $A = XDX^{-1}$, where $D = \sum_i \lambda_i |i\rangle\langle i|$.

[2] a polynomial is an expression of the form $p(x) = c_0 + c_1 x + \dots + c_k x^k$. The polynomial of the matrix A is another matrix of the form $p(A) = c_0 I + c_1 A + c_2 A^2 + \dots c_k A^k$.