Lecture & Proseminar 250078/250042
“Quantum Information, Quantum Computation, and Quantum Algorithms” WS 2024/25

— Exercise Sheet #2 —

Problem 1: The Bloch sphere

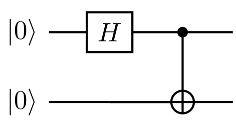
Recall that the Pauli matrices are denoted as $X = \sigma_1$, $Y = \sigma_2$ and $Z = \sigma_3$.

1. Given a state
$$|\psi\rangle = e^{i\chi}\big[\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\big] \tag{1}$$
show that
$$|\psi\rangle\langle\psi| = \tfrac{1}{2}(I + \vec{v}\cdot\vec{\sigma}) \ \text{ with } \vec{v}\in\mathbb{R}^3 \text{ and } |\vec{v}| = 1\ , \tag{2}$$
(i.e., $\vec{v}$ is a vector on the unit sphere in $\mathbb{R}^3$), where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$. (You should find that $\vec{v}$ is exactly the point on the Bloch sphere with spherical coordinates in θ and ϕ, just as introduced in the lecture.)
2. Show that the expectation value of the Pauli operators is $\langle\psi|\sigma_i|\psi\rangle = v_i$; i.e., $|\psi\rangle$ describes a spin which is polarized along the direction $\vec{v}$.
3. Show that for any state $|\psi\rangle$ with corresponding Bloch vector $\vec{v}$, the state $|\phi\rangle$ orthogonal to it, i.e. with $\langle\psi|\phi\rangle = 0$ (for qubits, i.e., in $\mathbb{C}^2$, this state is uniquely determined up to a phase!), is described by the Bloch vector $-\vec{v}$, i.e., it is located at the opposite point of the Bloch sphere.
4. *Bonus question:* Derive a general expression for the overlap $|\langle\phi|\psi\rangle|^2$ of two arbitrary states in terms of the corresponding Bloch vectors.

Note: A particularly elegant way to check 2) and 3) is to use that $\langle\psi|O|\psi\rangle = \mathrm{tr}[|\psi\rangle\langle\psi|\,O]$ together with Eq. (2) and $\mathrm{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$, but the results can of course also be derived directly from Eq. (1) with a bit more brute force.

Problem 2: Tensor Products

1. Write the column vectors representing the tensor products $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$.
2. Write the matrix form of the tensor product of all Pauli operators, i.e., $\sigma_i \otimes \sigma_j$.
3. What is the rank of $\mathbb{1}\otimes\mathbb{1} + X\otimes X + Y\otimes Y + Z\otimes Z$?
4. Write the matrix form of $H\otimes\mathbb{1}$, where $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ is the Hadamard gate.
5. Calculate $\mathrm{CNOT}\cdot(H\otimes\mathbb{1})|00\rangle$. Recall that $\mathrm{CNOT} = |0\rangle\langle 0|\otimes\mathbb{1} + |1\rangle\langle 1|\otimes X$. A pictorial representation of this calculation is



6. Using the previous calculation show that the CNOT gate is not a tensor product.

Problem 3: Unitary invariance

1. Show that the singlet state
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} - |10\rangle_{AB}\right)$$
is invariant under joint rotations by the same 2×2 unitary U with $det(U) = 1$, i.e.,
$$|\Psi^-\rangle = (U\otimes U)|\Psi^-\rangle$$
How does this formula change when $\det(U) \neq 1$?

2. Determine the states

$$\begin{aligned} &(X \otimes I)|\Psi^-\rangle \ , \qquad &(I \otimes X)|\Psi^-\rangle \ , \\ &(Y \otimes I)|\Psi^-\rangle \ , \qquad &(I \otimes Y)|\Psi^-\rangle \ , \\ &(Z \otimes I)|\Psi^-\rangle \ , \qquad &(I \otimes Z)|\Psi^-\rangle \ . \end{aligned}$$

In the light of point 1, why are they pairwise equal (up to global phases)?
Note: Together with $|\Psi^-\rangle$, these are known as the four *Bell states*.

3. Show that the maximally entangled state

$$|\Omega\rangle = \sum_{i=1}^{d} |i, i\rangle$$

of two qu-d-its (i.e., systems with a Hilbert space $\mathbb{C}^d$) is invariant under $U \otimes \bar{U}$, where U is any $d \times d$ unitary, that is,

$$|\Omega\rangle = (U \otimes \bar{U})|\Omega\rangle \ .$$