Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2024/25

— Exercise Sheet #3 —

Problem 1: Measurements

1. Find a measurement that differentiates between the states $\rho = \frac{1}{2}(I + aX)$ and $\sigma = \frac{1}{2}(I + bY)$, for any $a, b \in \mathbb{R}$, $|a| < 1$, $|b| < 1$, $a \neq 0$, $b \neq 0$. (That is, find a measurement that has different outcome probabilities in the two states.)
2. Find a measurement differentiating between $\rho = \frac{1}{2}(I + aX)$ and $\sigma = \frac{1}{2}(I + bX)$.
3. Give the outcome probabilities, post measurement states and the expectation value of measuring Y in the state
$$\rho = \frac{1}{2}(I + aX + bY + cZ).$$

Problem 2: Ensemble decompositions

In the following we construct different ensemble decompositions of the state
$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \quad p \in [0, 1].$$

1. Check that the above sum is already an ensemble decomposition of ρ.
2. Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Let
$$\sigma = \frac{1}{1-\mu}(\rho - \mu|+\rangle\langle +|).$$
For which values of μ is σ a density matrix? Check that for these values of μ the decomposition $\rho = \mu|+\rangle\langle +| + (1-\mu)\sigma$ is an ensemble decomposition of ρ.
3. For which value of μ is σ a pure state, i.e., $\sigma = |\psi\rangle\langle\psi|$? For this μ give the vector $|\psi\rangle$.
4. Check that for this value of μ there is a unitary matrix U such that
$$\sqrt{\mu}|+\rangle = U_{00}\sqrt{p}|0\rangle + U_{01}\sqrt{1-p}|1\rangle$$
$$\sqrt{1-\mu}|\psi\rangle = U_{10}\sqrt{p}|0\rangle + U_{11}\sqrt{1-p}|1\rangle.$$
5. Find an ensemble decomposition of ρ with three pure states.

Problem 3: Measurements and filtering

Suppose that a bipartite system AB is initially in the state
$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle\ .$$
The goal of Alice and Bob is to obtain a maximally entangled state
$$|\Omega\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
with some probability by applying local operations only. Specifically, the plan is that Alice will apply a POVM measurement to achieve that.

1. Show that the operators $M_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes I_B$ and $M_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes I_B$, with $0 \leq \gamma \leq 1$, define a POVM measurement. (Note that these describe measurements carried out on Alice's side only!)
2. Determine the outcome probabilities and the post-measurement states for both measurement outcomes.
3. Find a value γ such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.