Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2024/25

— Exercise Sheet #4 —

Problem 1: CPTP maps.

In this problem, we will study some commonly appearing CPTP maps (quantum channels). In addition to the problems listed, verify for each map that it is CPTP (completely positive trace preserving) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z\ .$$

Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z)\ ,$$

i.e., it acts as

$$\rho = \tfrac{1}{2}(I + r_x \cdot X + r_y \cdot Y + z_z \cdot Z) \mapsto \tfrac{1}{2}(I + (1-2p)r_x \cdot X + (1-2p)r_y \cdot Y + r_z \cdot Z).$$

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$M_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|\ ,$$

where $0 \leq \gamma \leq 1$. Here, M_0 describes a decay from $|1\rangle$ to $|0\rangle$, and γ corresponds to the decay rate.

   (a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

   where $0 \leq p \leq 1$ and η is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

   (b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter γ_1 and consider another amplitude damping channel $\mathcal{E}_2$ with parameter γ_2. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1-(1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the γ's as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z\ .$$

Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}I$.

Hint: Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.

Problem 2: Manipulation of entangled states.

During the lecture we have seen that the maximally entangled state $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$ has the property

$$(\mathbb{I} \otimes O)|\Omega\rangle = (O^T \otimes \mathbb{I})|\Omega\rangle,$$

for every $O \in \mathcal{B}(\mathcal{H})$. In this exercise we investigate some of the implications of this statement.

1. Show that if Alice and Bob share a maximally entangled state and Alice applies a CP map T (e.g. time evolution) while Bob doing nothing, then the result is the same as if Bob applied another CP map S on the state while Alice doing nothing:

$$(T \otimes \mathbb{I})(|\Omega\rangle\langle\Omega|) = (\mathbb{I} \otimes S)(|\Omega\rangle\langle\Omega|).$$

   How does T and S relate to each other? Assume that T is in addition trace preserving. What extra property does S have? Is it tace preserving?

2. Let Alice and Bob share a maximally entangled state and assume that Alice perform a POVM measurement described by measurement operators $\{M_i\}_{i\in I}$, and gets outcome i. Show that the corresponding post-measurement state can be obtained with the same probability if instead of Alice, Bob performs a suitable measurement on the maximally entangled state. Construct such a measurement.

Problem 3: SIC-POVMs

A *symmetric informationally complete POVM* (SIC-POVM) in d dimensions is given by a set of operators $\{F_i\}_{i=1}^{d^2} \subseteq \mathcal{B}(\mathbb{C}^d)$ of the form $F_i = \lambda \cdot |\phi_i\rangle\langle\phi_i|$, where $|\phi_i\rangle \in \mathcal{B}(\mathbb{C}^d)$ is normalized, $\||\phi_i\rangle\| = 1$, and

i) $\{F_i\}_{i=1}^{d^2}$ forms a POVM

ii) $\mathrm{tr}(F_iF_j) = K$ for $i \neq j$, independent of i and j (that is, the POVM is *symmetric*).

We will investigate SIC-POVMs.

1. Use the two conditions (i) and (ii) to determine the values of λ and K.

2. Now consider $d = 2$ (qubits). Consider four states $|\phi_i\rangle$ sitting at the four corners of a tetrahedron. (Any tetrahedron is good, but it might be convenient to have one corner along the z axis and another one in the x-z-plane.) Derive the form of $|\phi_i\rangle$, and show that they give rise to a SIC-POVM (following the convention above).

3. Show that the operators $\{F_i\}_{i=1}^{d^2}$ of a SIC-POVM (with the conditions (i) and (ii) above, for arbitary d) are linearly independent.

4. Show that the linear independence of the $\{F_i\}_{i=1}^{d^2}$ implies that there exist $\{K_i\}_{i=1}^{d^2}$ such that

$$\rho = \sum_{i=1}^{d^2} K_i \,\mathrm{tr}[F_i\rho], \quad \forall \rho \in \mathcal{B}(\mathbb{C}^d),$$

   that is, the POVM is *informationally complete*, i.e., we can reconstruct any state ρ from the outcome probabilities of the POVM. What is the form of the K_i?