Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2024/25

— Exercise Sheet #5 —

Problem 1: Decay of entanglement

Consider the maximally entangled state $\rho = |\Omega\rangle\langle\Omega|$, where $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like ρ are typically not stable, but decay over time. A typical evolution is that the off-diagonal elements decay relatively quickly to zero with a timescale T_2 ("dephasing"), while the diagonal elements become equal with a longer timescale T_1 ("decoherence"). Such an evolution is described as

$$\rho(t) = p_+|00\rangle\langle 00| + p_-|01\rangle\langle 01| + p_-|10\rangle\langle 10| + p_+|11\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|00\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|11\rangle\langle 00| \ ,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.

1. Give the matrix form of $\rho(t)$.
2. Determine the values of T_1 and T_2 for which $\rho(t) \geq 0$ for all times t. (You should find that T_2 cannot be much larger than T_1, otherwise $\rho(t)$ becomes unphysical – that is, there is indeed a natural reason why we would typically expect dephasing to occur on the faster timescale.)
3. What is the limit $\lim\limits_{t\to\infty} \rho(t)$? Is it entangled?
4. Take the partial transpose $\rho(t)^{T_B} = (\mathbb{I} \otimes T)(\rho(t))$ and give its matrix form.
5. Calculate the eigenvalues of $\rho(t)^{T_B}$. And sketch how they change over time for $T_1 = T_2 = 1$. What it the asymptotic limit?
6. Find the time t_{sep} after which $\rho(t_{\text{sep}})$ becomes separable.

Problem 2: Entanglement witnesses

Consider a bipartite system with $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$. Let $W := \mathbb{I} - d|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i,i\rangle$.

1. Show that W is an entanglement witness: $W \ngeq 0$ and $\text{tr}[W\rho] \geq 0$ for separable states ρ.
2. Consider the family
$$\rho_{\text{iso}}(\lambda) = \lambda\,\frac{\mathbb{I}}{d^2} + (1-\lambda)|\Omega\rangle\langle\Omega|$$
of *isotropic states*. In which range of λ is $\rho_{\text{iso}}(\lambda) \geq 0$? In which range of λ does W detect that $\rho_{\text{iso}}(\lambda)$ is entangled?
3. Consider the case $d = 2$. Does W detect the antisymmetric state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ as entangled? Generally, which property must a pure state satisfy to be detected by W?
4. Verify that $d \cdot (\Lambda \otimes \mathbb{I})(|\Omega\rangle\langle\Omega|) = W$ for the map Λ defined by $\Lambda(\rho) := d\,\text{tr}_B\big[W^T(\mathbb{I}_A \otimes \rho_B^T)\big]$. (That is, Λ maps to $\frac{1}{d}W$ through the Choi-Jamiolkowski isomorphism). What is Λ? Show that it is positive but not completely positive.
5. For a two-qubit system, in which range of λ does Λ detect that $\rho_{\text{iso}}(\lambda)$ is entangled? Does Λ detect the antisymmetric state?

Problem 3: Entanglement witness for a given state.

Construct an entanglement that detects the entanglement of a given pure entangled state $|\Psi\rangle$. For that, let $W = \alpha \cdot I - |\Psi\rangle\langle\Psi|$ and find a value of α (it can be implicitly given) for which W is entanglement witness detecting the entanglement of $|\Psi\rangle$.