Lecture & Proseminar 250078/250042
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2024/25

— Exercise Sheet #11 —

Problem 1: Quantum Error Correction Conditions for the 3-qubit code

Check the Quantum Error Correction Conditions

$$\langle \hat{\imath}|E_\alpha^\dagger E_\beta|\hat{\jmath}\rangle = c_{\alpha\beta}\delta_{ij} \tag{1}$$

for the 3-qubit bit flip code, and the error model with operators

$$E_0 = \sqrt{(1-p)^3}(I \otimes I \otimes I)\ ,$$
$$E_1 = \sqrt{(1-p)^2 p}(I \otimes I \otimes X)\ ,\ E_2 = \sqrt{(1-p)^2 p}(I \otimes X \otimes I)\ ,\ E_3 = \sqrt{(1-p)^2 p}(X \otimes I \otimes I)\ .$$

(Observe that the individual normalization of the E_α is not important for the correctness of Eq. (1), it only affects the specific $c_{\alpha\beta}$; in particular, we could have equivalently checked the Quantum Error Correction Conditions for $E_0 = I \otimes I \otimes I, E_1 = I \otimes I \otimes X, E_2 = I \otimes X \otimes I, E_3 = X \otimes I \otimes I$.)

Problem 2: Quantum Error Correction Conditions for the 9-qubit code

Also check the Quantum Error Correction Conditions for the 9-qubit code and arbitrary single-qubit errors, i.e., where $E_0 \propto I$, and $E_1, \dots, E_{27}$ are a single Pauli operator in any one position.
(This is not as tedious as it sounds, as you can use the structure of the code. One possibility is to work with the basis dual to the one given in the lecture, i.e. the sum and difference of the two basis states, and compare these two states $|\hat{+}\rangle$ and $|\hat{-}\rangle$ the computational basis – this quite directly gives the δ_{ij}, and it only remains to show that the prefactor is independent of i. But there are certainly other ways to use the structure.)

Problem 3: Rephrasing the Quantum Error Correction Conditions

Let $\mathcal{C}$ be a code and $\{E_\alpha\}$ an error model which is corrected by $\mathcal{C}$, i.e., for which the quantum error correction condition

$$\langle \hat{\imath}|E_\alpha^\dagger E_\beta|\hat{\jmath}\rangle = c_{\alpha\beta}\delta_{ij}\ \forall\, \alpha, \beta$$

holds for some ONB $|\hat{\imath}\rangle$ of $\mathcal{C}$.
Now let P be the orthogonal projector onto the codespace $\mathcal{C}$. Show that the quantum error correction condition is equivalent to the condition that

$$PE_\alpha^\dagger E_\beta P = c_{\alpha\beta} P\ \forall\, \alpha, \beta\ ,$$

i.e., any pair of noise operators $E_\alpha^\dagger E_\beta$ acts as the identity when projected back onto the code space.