Lecture & Proseminar 250121/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2025/26

— Exercise Sheet #1 —

Problem 1: Eigenvalues and Eigenvectors of Matrices.
Consider the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

1. Compute the eigenvalues and corresponding normalized eigenvectors of each matrix.
2. Verify that A, B, C are Hermitian.
3. Write each matrix in its diagonal (spectral) form

$$A = \sum_i \lambda_i \left|v_i\right\rangle \left\langle v_i\right|, \qquad B = \sum_i \mu_i \left|w_i\right\rangle \left\langle w_i\right|, \qquad C = \sum_i \nu_i \left|u_i\right\rangle \left\langle u_i\right|,$$

   where λ_i, μ_i, ν_i are the eigenvalues and $|v_i\rangle$, $|w_i\rangle$, $|u_i\rangle$ the corresponding eigenvectors. (In Problem 3, we will prove that any Hermitian matrix can indeed be expressed in this form.)
4. For each matrix A, B, C, find a unitary matrix U such that

$$U^\dagger X U = D,$$

   where $X \in \{A, B, C\}$ and D is diagonal with the corresponding eigenvalues on the diagonal.
5. Check that the eigenvectors of A, B and C form an orthonormal basis of $\mathbb{C}^d$.

Problem 2: Properties of Unitary Operators.

1. Show that unitaries preserve inner products: if $|\phi_i\rangle = U|\psi_i\rangle$ for $i = 1, 2$, then

$$\langle \phi_1 | \phi_2 \rangle = \langle \psi_1 | \psi_2 \rangle \ .$$

2. Show that unitaries preserve the norm induced by the Hilbert space, i.e.

$$\| U |\psi\rangle \| = \| |\psi\rangle \| \,.$$

3. If $U|\psi\rangle = \lambda|\psi\rangle$ (i.e. λ is an eigenvalue of U), show that $|\lambda| = 1$.
4. Show the converse statement: if U is unitarily diagonalizable[1] with eigenvalues satisfying $|\lambda| = 1$, then U is unitary.
5. Find a **non-diagonalizable** matrix with eigenvalues satisfying $|\lambda| = 1$ which is **not unitary**.

Problem 3: Spectral Decomposition of Hermitian Matrices.
Let A be a Hermitian $d \times d$ matrix.

1. Show that every eigenvalue λ_i of A is real. (Hint: consider the equation $A|\psi_i\rangle = \lambda_i|\psi_i\rangle$ and take the inner product with $\langle\psi_i|$.)

[1] i.e. U can be written as

$$U = \sum_{i=1}^{n} \lambda_i \left|\psi_i\right\rangle \left\langle \psi_i\right|$$

for an orthogonal basis $|\psi_i\rangle$

2. Show that eigenvectors corresponding to distinct eigenvalues are orthogonal, and that the set of eigenvectors can be chosen to form an orthonormal basis. (Hint: for degenerate eigenvalues, you can apply the Gram–Schmidt procedure to the corresponding subspace.)

3. Use the previous results to write A as an orthogonal decomposition:

$$A = \sum_{i=1}^{d} \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right| .$$

4. Define the unitary matrix U whose columns are the orthonormal eigenvectors $|\psi_i\rangle$, and the diagonal matrix $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$. Show that

$$A = UDU^\dagger .$$