Lecture & Proseminar 250121/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2025/26

— Exercise Sheet #4—

Problem 10: Ensemble decompositions by measurement.

a) Consider a state $|\psi\rangle = \alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B$, shared between two parties A and B, with Hilbert space dimensions $d_A = 2$ and $d_B = 4$, respectively. Determine the probabilities p_i and Alice's post-measurement states $|\phi_i\rangle$ if Bob measures in the basis (check that it is an ONB!)

$$(|0\rangle + |2\rangle)/\sqrt{2}\,, \quad (|1\rangle + |3\rangle)/\sqrt{2}\,, \quad (|0\rangle \pm |1\rangle - |2\rangle \mp |3\rangle)/2$$

(note the $\pm$).
What ensemble interpretation of Alice's state does this give? Check that this gives the correct reduced density matrix.

b) Consider the case where Bob's system has a general dimension d_B, and where he measures in a basis

$$|b_i\rangle = \sum u_{ij}|j\rangle\ .$$

i) What properties does the matrix $U = (u_{ij})$ satisfy?
ii) What is the form of the resulting post-measurement ensemble $\{(p_i, |\phi_i\rangle)\}$ for Alice's state?

Problem 11: Ambiguity of ensemble decomposition.

Complete the proof given in the lecture for the relation

$$\sqrt{p_i}|\psi_i\rangle = \sum u_{ij}\sqrt{q_j}|\phi_j\rangle$$

of different ensemble decompositions

$$\rho = \sum p_i|\psi_i\rangle\langle\psi_i| = \sum q_j|\phi_j\rangle\langle\phi_j|\ .$$

1. Show that for any ensemble decomposition $\rho = \sum q_i|\phi_i\rangle\langle\phi_i|$ with $q_i > 0\ \forall\, i$, it holds that $|\phi_i\rangle \in \mathrm{supp}(\rho)$. Here, $\mathrm{supp}(\rho)$ is the support of ρ as a linear map, that is, the orthogonal complement of its kernel $\ker(\rho)$. How does this justify the restriction $q_i \neq 0$ made in the lecture?
2. Show that any ensemble decomposition must have at least as many terms as the eigenvalue decomposition $\rho = \sum \lambda_k|e_k\rangle\langle e_k|$.
3. Show that the proof from the lecture extends to the case where the other decomposition has more terms than the eigenvalue decomposition, to show

$$\sqrt{p_i}|\psi_i\rangle = \sum u_{ik}\sqrt{\lambda_k}|e_k\rangle\ . \tag{$*$}$$

What property does this imply for $U = (u_{ik})$?
4. Show that that the relation $(*)$ can be inverted to give a formula for $\sqrt{\lambda_k}|e_k\rangle$.
5. Now consider the case where neither of the two ensembles is an eigenvalue decomposition. Use the fact that there are $U = (u_{ik})$ and $V = (v_{jk})$ which connect them to the eigenvalue decomposition to derive the general relation between two ensemble decompositions of a given state ρ. What is the form of the transformation matrix $W = (w_{ij})$ in terms of U and V? What properties do $W^\dagger W$ and $WW^\dagger$ satisfy?

Problem 12: Schmidt decomposition.

Find the Schmidt decomposition of the following states:

$$
\begin{aligned}
|\psi_1\rangle &= \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \\
|\psi_2\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2} \\
|\psi_4\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle}{2} \\
|\psi_5\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle}{2} \\
|\psi_6\rangle &= \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{3}}
\end{aligned}
$$