

Problem 13: Measurements and filtering.

Suppose that a bipartite system AB is initially in the state

$$|\phi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle.$$

The goal of Alice and Bob is to obtain a maximally entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

with some probability by applying local operations only. Specifically, the plan is that Alice will apply a POVM measurement to achieve that.

1. Show that the operators $M_0 = (|0\rangle\langle 0| + \sqrt{\gamma}|1\rangle\langle 1|)_A \otimes I_B$ and $M_1 = \sqrt{1-\gamma}|1\rangle\langle 1|_A \otimes I_B$, with $0 \leq \gamma \leq 1$, define a POVM measurement. (Note that these describe measurements carried out on Alice’s side only!)
2. Determine the outcome probabilities and the post-measurement states for both measurement outcomes.
3. Find a value γ such that one of post-measurement states becomes a maximally entangled state. Calculate the corresponding probability with which the initial state becomes a maximally entangled state.
4. In the lecture, we have shown that any POVM measurement can be implemented by adding an auxiliary system in state $|0\rangle$, applying a unitary, and measuring the auxiliary system in the computational basis. Construct such a unitary for the POVM of Alice above.

Problem 14: Unambiguous state discrimination.

Consider a set of linearly independent and normalized vectors $\mathcal{S} = \{|b_k\rangle\}_{k=1,\dots,d}$, $|b_k\rangle \in \mathbb{C}^d$, $\langle b_k|b_k\rangle = 1$ (i.e., a basis – but not an orthonormal one). In this problem, we will show that there is a POVM measurement which can unambiguously discriminate these states, that is, a POVM measurement

$$\{F_n\}_{n=1,\dots,d+1}$$

with $d+1$ outcomes $n = 1, \dots, d, d+1$, with the following property: If we are given an unknown state $|b_k\rangle \in \mathcal{S}$, then if the measurement returns some outcome $1 \leq n \leq d$, we know for sure that the state we have been given has been $|b_k\rangle$, while if we get the outcome $n = d+1$, we can’t make any conclusion (at least not with certainty) about which $|b_k\rangle$ we had been given. In other words, the test is able to unambiguously discriminate between the d states $|b_k\rangle$ (hence the name), but it has a chance of failing, in which case we learn that it failed.

1. Show that there is a dual basis $\{|c_k\rangle\}_{k=1,\dots,d}$ normalized vectors, $\langle c_k|c_k\rangle = 1$, such that

$$\langle c_k|b_\ell\rangle = \omega_k \delta_{kl} \tag{1}$$

(which again need not be orthonormal), with some constants $\omega_k > 0$.

(Hint: Make the ansatz $|c_k\rangle = \sum_j M_{kj}|b_j\rangle$ and insert this in (1).)

2. Choose $F_n = \lambda|c_n\rangle\langle c_n|$, $n = 1, \dots, d$, with a proportionality constant $\lambda > 0$. Show that for a suitable choice of λ , F_n can be completed to a POVM by adding a suitable F_{d+1} (that is, $F_n \geq 0$ for $n = 1, \dots, d+1$, and $\sum_{n=1}^{d+1} F_n = I$).

3. Determine the outcome probabilities of the POVM given the input $|b_k\rangle$,

$$p_n(|b_k\rangle\langle b_k|) = \text{tr}[F_n |b_k\rangle\langle b_k|] .$$

4. Work out the preceding steps – i.e., determine $\{|c_k\rangle\}$, the possible λ , and the $p_n(|b_k\rangle\langle b_k|)$ – for the concrete example of a qubit, $d = 2$, with $|b_1\rangle = |0\rangle$ and $|b_2\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, as a function of θ . What is the optimal choice for λ ?

Problem 15: SIC-POVMs.

A *symmetric informationally complete POVM* (SIC-POVM) in d dimensions is a POVM $\{F_i\}_{i=1,\dots,d^2}$ consisting of d^2 operators $F_i = \lambda\Pi_i$, where the $\Pi_i = |\phi_i\rangle\langle\phi_i|$ are rank-1 projectors $\Pi_i^2 = \Pi_i$, such that

- i) $\sum_{i=1}^{d^2} F_i = I$ (i.e. the F_i form a POVM), and

- ii) $\text{tr}(F_i F_j) = K$ for $i \neq j$, where K is independent of i and j (that is, the POVM is *symmetric*).

1. Use the two conditions (i) and (ii) to determine the values of λ and K .
2. Now consider $d = 2$ (qubits). Consider four states $|\phi_i\rangle$ sitting at the four corners of a tetrahedron. (Any tetrahedron is good, but it might be convenient to have one corner along the z axis and another one in the x - z -plane.) Derive the form of $|\phi_i\rangle$, and show that they give rise to a SIC-POVM (following the convention above).
3. Show that the operators $\{F_i\}$ of a SIC-POVM (with the conditions (i) and (ii) above, for arbitrary d) are linearly independent. (*Easier version:* Show this only for the qubit SIC-POVM constructed in point 2.)
4. Show that the linear independence of the $\{F_i\}$ implies that there exist K_i such that we can write

$$\rho = \sum_{i=1}^{d^2} G_i \text{tr}[K_i \rho]$$

– that is, the POVM is *informationally complete*, i.e., we can reconstruct any state ρ from the outcome probabilities of the POVM. What is the form of the K_i ?