**Lecture & Proseminar 250121/250122**
**"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2025/26**

**— Exercise Sheet #6—**

## Problem 16: Quantum channels.

In this problem, we will study some commonly appearing quantum channels. In addition to the problems listed, verify for each channel that it is a CPTP map (completely positive trace preserving map) and give its Kraus representation.

1. *Dephasing channel.* This channel acts as

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\,Z\rho Z \ .$$

   Show that the action of the dephasing channel on the Bloch vector is

$$(r_x, r_y, r_z) \mapsto ((1-2p)r_x, (1-2p)r_y, r_z) \ ,$$

   i.e., it preserves the component of the Bloch vector in the $Z$ direction, while shrinking the $X$ and $Y$ component.

2. *Amplitude damping channel.* The amplitude damping channel is giving by the Kraus operators

$$M_0 = \sqrt{\gamma}|0\rangle\langle 1|, \quad M_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1| \ ,$$

   where $0 \leq \gamma \leq 1$. Here, $M_0$ describes a decay from $|1\rangle$ to $|0\rangle$, and $\gamma$ corresponds to the decay rate.

   (a) Consider a single-qubit density operator with the following matrix representation with respect to the computation basis

$$\rho = \begin{pmatrix} 1-p & \eta \\ \eta^* & p \end{pmatrix},$$

   where $0 \leq p \leq 1$ and $\eta$ is some complex number. Find the matrix representation of this density operator after the action of the amplitude damping channel.

   (b) Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{E}_1$ with parameter $\gamma_1$ and consider another amplitude damping channel $\mathcal{E}_2$ with parameter $\gamma_2$. Show that the composition of the channels, $\mathcal{E} = \mathcal{E}_1 \circ \mathcal{E}_2$, $\mathcal{E}(\rho) = \mathcal{E}_1(\mathcal{E}_2(\rho))$, is an amplitude damping channel with parameter $1-(1-\gamma_1)(1-\gamma_2)$. Interpret this result in light of the interpretation of the $\gamma$'s as a decay probability.

3. *Twirling operation.* Twirling is the process of applying a random Pauli operator (including the identity) with equal probability. Explain why this corresponds to the channel

$$\mathcal{E}(\rho) = \tfrac{1}{4}\rho + \tfrac{1}{4}X\rho X + \tfrac{1}{4}Y\rho Y + \tfrac{1}{4}Z\rho Z \ .$$

   Show that the output of this channel is the maximally mixed state for any input, $\mathcal{E}(\rho) = \frac{1}{2}I$.

   *Hint:* Represent the density operator as $\rho = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$ and apply the commutation rules of the Pauli operators.

## Problem 17: CHSH inequality – Tsirelson's bound.

Tsirelson's inequality bounds the largest possible violation of the CHSH inequality $|\langle C\rangle| \leq 2$, with

$$\langle C\rangle = \langle a_0 b_0\rangle + \langle a_1 b_0\rangle + \langle a_0 b_1\rangle - \langle a_1 b_1\rangle \ . \tag{1}$$

in quantum mechanics – namely $|\langle C\rangle| = 2\sqrt{2}$). To this end, let $a_0, a_1, b_0, b_1$ be Hermitian operators (on some finite-dimensional complex Hilbert space) with eigenvalues $\pm 1$, so that

$$a_0^2 = a_1^2 = b_0^2 = b_1^2 = I \ .$$

Here, $a_0$ and $a_1$ describe the two measurements of Alice, and $b_0$ and $b_1$ those of Bob; in particular, this means that Alice's and Bob's measurements commute, i.e. $[a_x, b_y] = 0$ for all $x, y = 0, 1$. Define

$$C = a_0 b_0 + a_1 b_0 + a_0 b_1 - a_1 b_1 \ .$$

1. Determine $C^2$.
2. The operator norm of a bounded operator $M$ is defined by

   $$\|M\| = \sup_{|\psi\rangle} \frac{\|M|\psi\rangle\|}{\||\psi\rangle\|} \ .$$

   Verify that the operator norm has the properties

   $$\begin{aligned} \|MN\| &\leq \|M\|\|N\| \ , \\ \|M + N\| &\leq \|M\| + \|N\| \ . \end{aligned} \tag{2}$$

3. Show that $\|M\|$ equals the maximum eigenvalue of $\sqrt{M^\dagger M}$ (and thus, the largest singular value of $M$). What is then the operator norm of a hermitian operator in terms of its eigenvalues?
4. Find an upper bound on the norm $\|C^2\|$ (using the inequalities (2)).
5. Show that for Hermitian operators $\|C^2\| = \|C\|^2$. Use this to obtain an upper bound on $\|C\|$.
6. Explain how this inequality gives a bound on the maximum possible violation of the CHSH inequality in quantum mechanics. This is known as Tsirelson's bound, or Tsirelson's inequality.