Lecture & Proseminar 250121/250122
"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2025/26

— Exercise Sheet #7—

Problem 18: Teleportation-inspired protocols.

In this problem, we will get to know two variants of the teleportation protocol.

Part 1: Gate teleportation.

Gate teleportation is a variation of quantum teleportation that is being used in fault-tolerant quantum computation (a topic which will be covered later in the course of the lecture).
Suppose that we would like to perform a single-qubit gate (i.e., unitary) U on a qubit in state $|\psi\rangle$, but the gate is difficult to perform – e.g., it might fail and thereby destroy the state on which we act on. On the other hand, the gate $U\sigma_j U^\dagger$, where σ_j is any one of the three Pauli matrices, is easy to perform.

1. Verify that such a situation is given when the difficult operation is $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{smallmatrix}\right)$, while Paulis and $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$ are easy to realize.

2. Consider the following protocol to implement U on a state $|\psi\rangle_{A'}$:

   - Prepare $|\chi\rangle_{AB} = (I_A \otimes U_B)|\Phi^+\rangle_{AB}$, with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. ($U_B$ is still hard to implement, but we can try as many times as we want without breaking anything.)
   - Perform a measurement of $A'A$ in the Bell basis (A' is the register used to store $|\psi\rangle_{A'}$).
   - Depending on the measurement outcome, apply $U\sigma_j U^\dagger$ on the B system.

   Show that this protocol works as it should – that is, it yields the state $U|\psi\rangle$ in the B register with unit probability.

Part 2: Remote state preparation.

Remote state preparation is another variation on the teleportation protocol. In the variant we consider here, Alice has a *classical description* of a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ (on the equator of the Bloch sphere), i.e., she knows ϕ. The task is to prepare the state $|\psi\rangle$ on Bob's side, without Bob learning anything about ϕ.
To this end, let Alice and Bob share a maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

1. Find a state $|\chi\rangle$ such that when Alice's part of $|\Phi^+\rangle$ is projected onto $|\chi\rangle$, Bob is left with $|\psi\rangle$.

2. Now let Alice perform a measurement in the basis $\{|\chi\rangle, |\chi^\perp\rangle\}$, where $|\chi^\perp\rangle$ is the state perpendicular to $|\chi\rangle$ (since the space is 2-dimensional, $|\chi^\perp\rangle$ is unique up to a phase). Determine the post-measurement state of Bob for both of Alice's outcomes.

3. Show that if Alice communicates one bit to Bob, and Bob performs an operation which depends on this bit (which information is in the bit? what operation does Bob have to perform?), then Bob recovers $|\psi\rangle$ with unit probability.

4. A more "direct" way – given we know the protocol for teleportation – for Alice and Bob to realize the remote state preparation protocol would have been that Alice prepares $|\psi\rangle$ and then teleports it to Bob. Is there a way to relate these two protocols? How can the remote state preparation protocol be interpreted in terms of teleportation? In particular, in the teleportation protocol, Alice would have had to send *two* bits to Bob – what happened to the second bit?

Problem 19: Teleportation for qudits.

In this problem, we will study teleportation for qudits, i.e., quantum systems with a d-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$.
Let $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k, k\rangle$ be the maximally entangled two-qudit state (the generalization of $|\phi^+\rangle$ from the lecture), and define the two matrices

$$X = \sum_{k=0}^{d-1} |k+1\rangle\langle k| \ , \quad Z = \sum_{k=0}^{d-1} \omega^k |k\rangle\langle k| \ ,$$

with $\omega = e^{2\pi i/d}$, and with the convention that addition is modulo d (i.e., $|d\rangle\langle d-1| = |0\rangle\langle d-1|$).

1. Write X and Z in matrix notation. Show that for $d = 2$, X and Z reduce to the corresponding Pauli matrices.
2. Show that X and Z are unitary.
3. Show that $XZ = \eta ZX$, $\eta \in \mathbb{C}$. What is η?
4. Show that the d^2 states

$$|\psi_{\alpha\beta}\rangle = (X^\alpha Z^\beta \otimes I)|\Omega\rangle \ , \quad \alpha, \beta = 0, \dots, d-1 \ , \tag{1}$$

   are all maximally entangled states of two qudits (i.e., they have Schmidt rank d and all Schmidt coefficients are equal).
5. Show that the $|\psi_{\alpha\beta}\rangle$ form an orthonormal basis of $\mathbb{C}^d \otimes \mathbb{C}^d$.
6. Show that the X and Z in the definition (1) can be moved to the other side of tensor product, i.e.,

$$|\psi_{\alpha\beta}\rangle = (I \otimes Z^{\beta'} X^{\alpha'})|\Omega\rangle \ ,$$

   and determine how α' and β' depend on α and β.
7. Show that the teleportation protocol discussed in the lecture can be generalized to teleport an unknown qudit state $|\chi\rangle_{A'} \in \mathbb{C}^d$ from Alice to Bob, given that Alice and Bob share a maximally entangled state $|\Omega\rangle_{AB}$, where Alice measures A and A' in the basis $\big\{|\psi_{\alpha\beta}\rangle\big\}_{\alpha,\beta=0}^{d-1}$. Which information does Alice have to send to Bob, and what correction operation does he have to apply?

In fact, teleportation does not require to measure in the Bell basis or its generalization (1) above, but a measurement in any orthonormal basis $\{|\phi_b\rangle\}_{b=1}^{d^2}$ of maximally entangled states – i.e., where *all* $|\phi_b\rangle$ are maximally entangled – will work, as we will see in the following. (*Note:* If you want, you can prove this part of the problem first – then, you can essentially skip the steps above from Step 6 onwards.)
We will focus our discussion on a *single* maximally entangled state $|\phi\rangle$ – this could be one of the states from the basis $\{|\psi_b\rangle\}_b$, but this does not need to be the case: We will simply study what happens if Alice performs a projective measurement which *contains* the projection onto $|\phi\rangle$ as one measurement outcome, and we will study what happens when Alice gets this outcome, i.e., when she has projected her systems AA' onto $|\phi\rangle$.

1. Show that (since it is a maximally entangled state), $|\phi\rangle$ can be written as

$$|\phi\rangle = (U \otimes I)|\Omega\rangle = (I \otimes U^T)|\Omega\rangle$$

   with U unitary (cf. Problem 6).
2. Assume Alice has projected her systems AA' onto $|\phi\rangle$. Show that in that case, Bob can apply a correction operation (which?) to recover the unknown state $|\chi\rangle$, i.e., the state has been successfully teleported to Bob.