Lecture & Proseminar 250121/250122
“Quantum Information, Quantum Computation, and Quantum Algorithms” WS 2025/26

— Exercise Sheet #9—

Problem 23: Decay of entanglement.

Consider a Bell state $\rho = |\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Superposition states like ρ are typically not stable, but decay over time. A typical evolution is that the off-diagonal elements decay relatively quickly to zero with a timescale T_2 (“dephasing”), while the diagonal elements become equal with a longer timescale T_1 (“decoherence”). Since such decay processes converge exponentially, the state thus evolves as

$$\rho(t) = p_+|00\rangle\langle 00| + p_-|01\rangle\langle 01| + p_-|10\rangle\langle 10| + p_+|11\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|00\rangle\langle 11| + \tfrac{1}{2}e^{-t/T_2}|11\rangle\langle 00| ,$$

with $p_\pm = \frac{1}{4}(1 \pm e^{-t/T_1})$.

1. Give the matrix form of $\rho(t)$.
2. Determine the values of T_1 and T_2 for which $\rho(t) \geq 0$ for all times t. (You should find that T_2 cannot be much larger than T_1, otherwise $\rho(t)$ becomes unphysical – that is, there is indeed a natural reason why we would typically expect dephasing to occur on the faster timescale.)
3. What is the limit $\lim_{t\to\infty} \rho(t)$? Is it entangled?
4. Take the partial transpose $\rho(t)^{T_B}$ and give its matrix form.
5. Calculate the eigenvalues of $\rho(t)^{T_B}$.
6. Sketch how the eigenvalues change over time for $T_1 = T_2 = 1$. What it the asymptotic limit?
7. Find the time t_{sep} after which $\rho(t_{\text{sep}})$ becomes separable.

Problem 24: Bell inequalities and witnesses.

The CHSH operator – that is, the operator measured in the CHSH inequality – can be written as

$$C = \vec{n}_1\vec{\sigma} \otimes \vec{n}_0\vec{\sigma} + \vec{n}_1\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} + \vec{n}_3\vec{\sigma} \otimes \vec{n}_2\vec{\sigma} - \vec{n}_3\vec{\sigma} \otimes \vec{n}_0\vec{\sigma}$$

with $\vec{n}_k = (\cos(k\pi/4), 0, \sin(k\pi/4))$. Then, the CHSH inequality states that $|\text{tr}[C\rho]| \leq 2$ for all ρ which are described by a local hidden variable (LHV) model.

1. Show that the measurement of C on any separable state $\rho = \sum p_i \rho_i^A \otimes \rho_i^B$ can be described by an LHV model.
2. Use C to construct an entanglement witness W. (The witness should return $\text{tr}[W\rho] < 0$ exactly if ρ violates the CHSHS inequality.) Provide an explicit form of the witness.
3. In which range of λ does this witness detect Werner states $\rho(\lambda) = \lambda|\Psi^-\rangle\langle\Psi^-| + \frac{1-\lambda}{4}I$, with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$? How does it compare to the entanglement witness $W = \mathbb{F}$ discussed in the lecture?

Problem 25: Witnesses and the reduction criterion.

Consider a bipartite system with $\dim \mathcal{H}_A = \dim \mathcal{H}_B$. Let $W := \mathbb{I} - d|\Omega\rangle\langle\Omega|$, with $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle$.

1. Show that $\mathrm{tr}[W\rho] \geq 0$ for separable states ρ, i.e., W is an entanglement witness.
2. Consider the family
$$\rho_{\mathrm{iso}}(\lambda) = \lambda \frac{\mathbb{I}}{d^2} + (1-\lambda)|\Omega\rangle\langle\Omega|$$
of *isotropic states*. In which range of λ is $\rho_{\mathrm{iso}}(\lambda) \geq 0$? In which range of λ does W detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled?
3. Consider the case $d = 2$. Does W detect the antisymmetric state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ as entangled? Generally, which property must a pure state satisfy to be detected by W?
4. Consider the positive map $\Lambda(\rho) := d\,\mathrm{tr}_B\left[W^T(\mathbb{I}_A \otimes \rho_B^T)\right]$ (this is the map corresponding to the "Choi state" W^T via the reverse direction of the Choi-Jamiolkowski isomorphism). Determine the explicit form of Λ, and prove that it is a positive map. (Note that it cannot be completely positive, as its "Choi state" is W, which is not a state.)
5. For a two-qubit system, in which range of λ does Λ detect that $\rho_{\mathrm{iso}}(\lambda)$ is entangled? Does Λ detect the antisymmetric state?

(*Note:* The corresponding criterion for entanglement – i.e., when $(\Lambda \otimes \mathbb{I})(\rho) \not\geq 0$ – is called the *reduction criterion*. The name hopefully makes sense if you consider the explicit form of $\Lambda \otimes \mathbb{I}$.)