**Lecture & Proseminar 250121/250122**
**"Quantum Information, Quantum Computation, and Quantum Algorithms" WS 2025/26**

**— Exercise Sheet #12 —**

### Problem 34: Working with the 3-qubit bit flip code.

In this problem, we will study how to work with the 3-qubit bit flip code, i.e., how to explicitly perform the error correction, and also look at how to implement some gates without decoding the information.

1. Consider a qubit encoded with the 3-qubit code. Find a circuit which measures the error syndrome (i.e. which of the three qubits, if any, differs from the others), consisting of elementary gates and single-qubit measurements in the computational basis, and possibly using ancillas in the $|0\rangle$ state. (You should only need CNOT gates.) For each measurement outcome, give the correction operation.
2. Show that instead of measuring the ancillas, we can also perform quantum gates for the correction, and then discard (trace out) the ancillas, without the need for a measurement. Can this also be done only with CNOTs and simple single-qubit gates (Hadamard, Pauli)?
3. Show that the Pauli operators on the encoded (logical) qubit can be implemented by acting with single-qubit gates on the physical qubits, without decoding the code. (Again, single-qubit Paulis should suffice.)
4. Now consider two qubits, each encoded with a 3-qubit code. What happens when we apply CNOT gates between all three pairs of physical qubit (i.e. between qubit 1 of the 1st qubit and qubit 1 of the 2nd qubit, etc.)? (Logical gates which can be implemented in this way are called *transversal gates*; note that the same property also holds for the Paulis above.)

### Problem 35: Quantum Error Correction Conditions for the 3-qubit code.

Check the Quantum Error Correction Conditions

$$\langle\hat{\imath}|E_\alpha^\dagger E_\beta|\hat{\jmath}\rangle = c_{\alpha\beta}\delta_{ij} \tag{1}$$

for the 3-qubit bit flip code, and the error model with operators

$$E_0 = \sqrt{(1-p)^3}(I\otimes I\otimes I)\ ,$$
$$E_1 = \sqrt{(1-p)^2p}(I\otimes I\otimes X)\ ,\ E_2 = \sqrt{(1-p)^2p}(I\otimes X\otimes I)\ ,\ E_3 = \sqrt{(1-p)^2p}(X\otimes I\otimes I)\ .$$

(Observe that the individual normalization of the $E_\alpha$ is not important for the correctness of Eq. (1), it only affects the specific $c_{\alpha\beta}$; in particular, we could have equivalently checked the Quantum Error Correction Conditions for $E_0 = I\otimes I\otimes I, E_1 = I\otimes I\otimes X, E_2 = I\otimes X\otimes I, E_3 = X\otimes I\otimes I$.)

### Problem 36: Rephrasing the Quantum Error Correction Conditions.

Let $\mathcal{C}$ be a code and $\{E_\alpha\}$ an error model which is corrected by $\mathcal{C}$, i.e., for which the quantum error correction condition

$$\langle\hat{\imath}|E_\alpha^\dagger E_\beta|\hat{\jmath}\rangle = c_{\alpha\beta}\delta_{ij}\ \forall\,\alpha,\beta$$

holds for some ONB $|\hat{\imath}\rangle$ of $\mathcal{C}$.

Now let $P$ be the orthogonal projector onto the codespace $\mathcal{C}$. Show that the quantum error correction condition is equivalent to the condition that

$$PE_\alpha^\dagger E_\beta P = c_{\alpha\beta}P\ \forall\,\alpha,\beta\ ,$$

i.e., any pair of noise operators $E_\alpha^\dagger E_\beta$ acts as the identity when projected back onto the code space.