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Symmetries and excitations in the quantum double models - a tensor network approach

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1 Introduction

1.1 Motivation

The aim of this thesis is to study particular topologically ordered systems by tensor networks.
A *tensor network* is nothing but a contraction of the different indices of multiple different tensors. As quantum (and in general, statistical) physics is linear, tensors (and with them, tensor networks) are ubiquitous there. One of the most fundamental principles of physics is locality. A tensor network representing a state or observable (which are themselves tensors) that respects the locality structure of the physical space is called *tensor network state* or *tensor network operator*. By this we mean a tensor network that has tensors that are locally spread over the physical space, and that have each one or few (*physical*) indices that correspond the local indices of the final resulting tensor. In addition, they each have one or few (*virtual*) indices that are contracted locally between neighboring tensors. The resulting states are also known as MPS [9] in one dimension or PEPS [8] in two or more dimensions. It is conjectured (and partly proven for different situations) that such tensor network states approximate efficiently thermal and ground states of local Hamiltonians. Therefore, they can be used as faithful representations of states and observables in variational algorithms, like the famous DMRG [10]. Apart from that numeric use, tensor networks are also a tool for the analytic study of many-body systems. Properties like the entanglement structure or the satisfaction of area laws have a nice interpretation in the language of tensor network states. Another central result that builds upon MPS is the classification of all quantum phases in one dimension [12, 16]. Also topologically ordered systems have been found to have a characterization in terms of PEPS.
Topological order is a new kind of order that goes beyond the Landau theory of phases. In topologically ordered systems, operators creating excitations, as well as order parameters have the form of strings. Other characteristic properties are the ground state degeneracy depending on the topology of the space, the topological entanglement entropy which is a constant correction to the area law, as well as the existence of long-range entanglement that is not reflected in two-point correlation functions. There are different reasons why one would want to study topologically ordered systems. First, they are a candidate for building quantum memory, incorporating a mechanism of quantum error correction at the physical level [5]. To this end we use that the fact that the ground state space on a topologically non-trivial surface, or the space containing multiple particles, has some extra degeneracy. This additional degree of freedom is completely non-local and can be used to store the quantum information. This degree of freedom can also be processed by moving particles around and braiding them with each other. Measurements can be done by fusion of pairs of particles and observation of the outcome. It was shown that the corresponding degeneracy is stable under arbitrary local perturbations [15]. However, thermal

fluctuations lead to problems with this stability [7]. Second, they explain a variety of phenomena that occur in the context of new, exotic phases of matter in condensed matter systems, like the fractional quantum Hall effect [11]. Finally, they also open a new perspective to questions of fundamental quantum physics. For example, via the mechanism of string-net condensation, fermionic degrees of freedom (together with gauge fields) can be modeled by nothing but local bosonic degrees of freedom [6].
In the PEPS representation, topological order manifests itself in terms of a symmetry acting only on the virtual legs of the tensor network and that is stable under blocking. The symmetry can be implemented by the common action of a group to all virtual legs [1], or more generally by matrix product operator algebras [3]. This virtual symmetry around a big blocked region has the form of a string operator in the virtual level. Those strings around non-contractible loops of some topologically non-trivial surface parametrise different ground states, and open strings carry excitations at their endpoints.
In this thesis, we investigate those tensor network representations for the quantum double models [5]. We generalize some of the results obtained in [1] to the quantum doubles for non-abelian groups and arbitrary Hopf*-algebras. To this end, we start with the the local physical symmetries of the RG fixed-point model to derive different tensor network representations of the ground states and low-lying excitations. We will see that all arising PEPS-tensors will also have virtual symmetries of the same kind as the physical ones.

1.2 Overview

In Sec.(2), we introduce the necessary algebraic structures that describe symmetries for the quantum double models. We first motivate why a generalization of groups to a (quantum) statistical theory naturally leads to Hopf*-algebras and their representations. To this end, we introduce a graphical tensor-network language to reveal the beauty of the corresponding structures.
Sec.(3) explains in a very general framework how we can use local, commuting symmetries to define local excitations in our system, making use of the algebraic structures introduced in Sec.(2). It is also shown how to generally derive creation operators for those particles, and the different possible topological natures of those excitations are discussed.
Sec.(4) begins by defining the quantum double models on arbitrary lattices and for arbitrary Hopf*-algebras. First, we examine the commutativity properties of the defined symmetries, and show how the quantum double arises as a symmetry. Then we write the global symmetry (including ground state projectors at most places) as tensor network, and use this to calculate the topological degeneracy for different surfaces and charge configurations. From this global representation we derive tensor networks for the ground states and excitations themselves. We put those tensor networks in different forms and investigate the virtual symmetries of the corresponding PEPS-tensors. Then the transfer operator and its fixed points are constructed and its symmetries are used to define a

physical system at the virtual boundary. The virtual strings are identified with the operators creating local excitations of this boundary system. In the end, we also look at an example of symmetry breaking with charge condensation and confinement from our symmetry perspective.
All statements in the main text hold for general Hopf*-algebras, whereas some special results obtained for group algebras are in the examples or in the appendix. The latter contains the diagonalisation of quantum doubles of groups, one specific example for the letter procedure, and an instruction how to build the corresponding virtual strings.

2 Symmetries: from groups to Hopf*-algebras

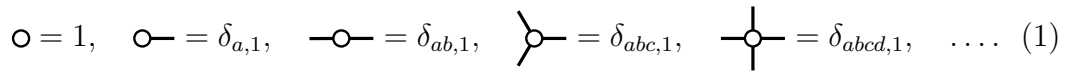
2.1 Notational conventions

During this thesis, we will make extensive use of a graphical notation for tensor networks called (*tensor network notation*). In this notation, a tensor is depicted by some symbol/shape and each index of the tensor corresponds to a line (also called leg) that comes out of the corresponding shape. The tensor product of two or more tensors is represented by simply drawing them near each other on the same piece of paper. When two open indices coming out of different or the same tensor are connected, this denotes their *contraction*, so the summation over the tensor entries where the index values are the same.
In some places we use a slight generalization of the common tensor network language: The dimension of an index might depend on the value of another index. This way, we can also draw direct sums of tensors of different dimension. Often different line styles indicate different types of dimension spaces of the corresponding indices. Lines might be merged to one index with the product of their dimensions, or conversely split up. The elementary tensors that are used to build up tensor networks have some shape (or frame) that has some specific form, and some content. The shape corresponds to the function or properties that the tensor has. The content of the tensor is a specific label, and sometimes also the labels of other tensors with respect to those it has some specific function or property. This labels can be anything, like symbols, patterns, colors, or blank. Tensors might be drawn in rotated form without changing their meaning, but they are never flipped. When an index line crosses another one or crosses over some tensor, there is no effect. Lines and the frames of shapes might be colored to highlight or distinguish parts of the network, without changing the network itself, whereas changing the color of the content means changing the tensor. If an equation between two tensor networks is given, the open indices on both sides are identified with each other by the following criteria: First, the line styles (or types) have to match. Second, the position of the line end or the direction in which it points out of the network is often approximately the same. Third, if the identification is still unclear, additional labels or markers (like arrows) are added to the line ends on both sides. If line ends are dressed with three dots, this means that we're looking only at some cut-out part of a bigger tensor network. Those dotted indices do not have to match in an equation, e.g. the number of contractions between the cut-out part and the rest can change.

2.2 Groups and *-algebras

In classical deterministic systems, a symmetry is given by a group $\mathcal{G}$ together with an action of $\mathcal{G}$ as permutations over the set of configurations, leaving invariant the physical laws. If we go over to some statistical (classical or quantum) description of nature, physical states are probability distributions or density

matrices, i.e. linear structures as generalization of the discrete configuration set. It is thus natural to also replace the discrete structures implementing the notion of symmetry in the deterministic case by linear structures, so superpositions of the deterministic structures. This will lead us from groups and their action via permutations to *Hopf*-algebras* and their linear representations.
Let us first examine further the classical, deterministic case: A group $\mathcal{G}$ over a set G is determined by three objects: the *unit* $1 \in G$, the *inverse* (which we will also call *involution*) $* \in G^G$, and the *product* $\cdot \in G^{G\times G}$, satisfying a set of axioms. For aesthetic reasons, we do not consider the product, but instead the product joined with the involution $a, b \to (ab)^{-1}$. We can represent these three objects in a linearised form as tensors in $\mathbb{C}^G$, $\mathbb{C}^{G\times G}$, $\mathbb{C}^{G\times G\times G}$. To this end we replace it by a tensor with one index for each slot (i.e. input or output variable) of the object which runs over the elements of the set corresponding to the slot. The components of the tensor are then taken to be 1 if the values of the incoming and outgoing indices fit to each other, and 0 otherwise:



Remark 1. In Eq.(1), the open indices of the tensors on the left have G as a dimension set and correspond in counterclockwise sense to the indices $a, b, c, d, \ldots$ on the right. Note that it doesn't matter where we begin with a, whereas the order or indices matters (for non-abelian groups). The linearised version of the unit, the involution and the product (joined with involution) correspond to the 1, 2, and 3-leg tensors, respectively. Even they are different tensors, they all have a very similar structure, and we can also define tensors with 0, 4 or more legs with the same structure. That's why we denote them all with the same symbol (a circle with blank content) and only distinguish them by their number of legs.

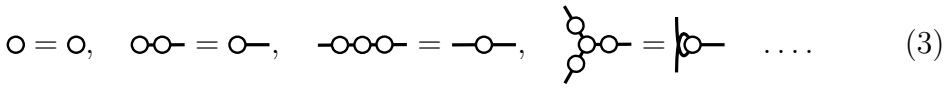
In this linearised language, the group axioms become tensor equations. To reveal their structure, we formulate these axioms for the set of all tensors with arbitrary many legs, not just for the 1, 2, and 3-leg tensors. We can then choose axioms that are equivalent to the usual group axioms and group them into sets, such that the axioms contained in one set have a very similar structure and can be parametrised by one or two numbers:

Cyclicity axioms:

These axioms just mean that the tensors are invariant under cyclic permutations of the indices. They are already implicitly contained in the notation: The symbol for the tensors is rotation symmetric, such that there's no way to determine where to start. The first two equations are trivially fulfilled, the second one says that $*$ is hermitian, the third one is equivalent to $abc = 1 \Leftrightarrow cab = 1$.

Apart from cyclic permutations, the order of indices matters (at least for non-abelian groups). Especially, one cannot flip any of the tensors. When doing so, one has to invert all indices due to the

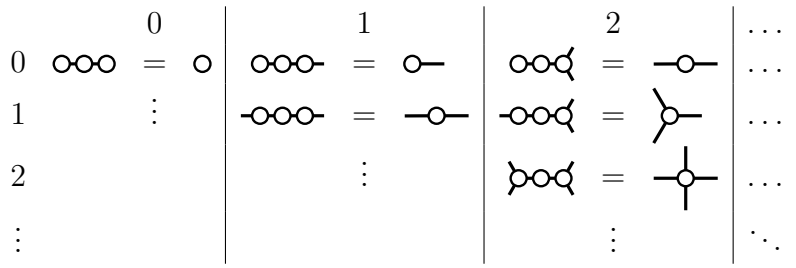
Flip axioms:



(3)

The first axiom is equivalent to $1^{-1} = 1$, the second means (almost) that $*$ is an involution, and the third one corresponds to the $*$-property $(ab)^{-1} = b^{-1}a^{-1}$. The last group of axioms are the

Fusion axioms:



(4)

These axioms correspond to

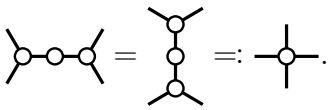
$$(ab\cdots)(xy\cdots) = 1 \Leftrightarrow ab\cdots xy\cdots = 1.$$

Strictly speaking, the fact that the 2-leg tensor (inverse) is an involution,

(5)

does not follow from the axioms Eq.(2), Eq.(3), Eq.(4), so we have to add it to our axioms as well.

Remark 2. The cyclicity of the 4-leg symbol together with the fusion rule for two 3-leg symbols to one 4-leg symbol together lead to the *associativity* condition:



(6)

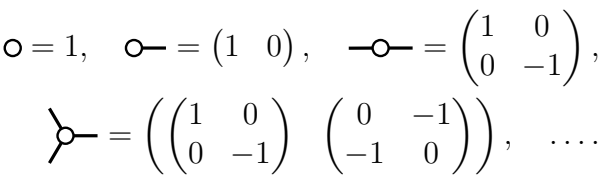
Remark 3. The group is actually fully determined by the 1, 2 and 3-leg tensors fulfilling only the corresponding subset of the axioms together with the associativity condition. The fusion axioms Eq.(4) then contain the definition for the tensors of order 0 and higher than 3.

So far we have only been describing classical groups in a language that is complete overkill for this setting. But now we are at a point where we can actually linearise the notion of a group to get something more general. To this

end we can simply allow general sets of 0, 1, 2, 3-leg and higher order tensors fulfilling the same axioms. This leads us to the definition of the following structure that we consider the linearised generalization of the unit, inverse, and product of a group:

Definition 1. We will call a **-algebra* a set of tensors with 0, 1, 2, 3, ... legs, that fulfill the axioms Eqn.(2),(3),(4),(5). The whole set is actually already determined by the 1, 2 and 3-leg tensor. The 1-leg tensor is also called *unit*, the 2-leg tensor *involution*, and the three-leg tensor regarded as a map from two to one indices, joined with the involution is called *multiplication*.

Proposition 1. The *complex numbers* are nothing but a two-dimensional real *-algebra, if we define the tensors above like:



(7)

From now on, we will only deal with complex *-algebras (which we will call *C*-algebras*), i.e. *-algebras that are a tensor product of some real *-algebra with the complex numbers in Eq.(7). So we will treat the complex nature of the algebras via an additional two-dimensional real index that will often be hidden. However, what we will call the tensor product of two C*-algebras does not contain two copies of that two-dimensional algebra, but will be the complexification of the tensor product of the two corresponding real algebras.

Remark 4. Our definition of a *-algebra turns out to be equivalent in finite dimensions to what is known as *unital associative *-algebra*. However, in the usual language of *-algebras there is no notion of cyclicity (Eq.(2)) of the product. But it is a central result in the theory of C*-algebras that every C*-algebra is isomorphic to a direct sum of full complex matrix algebras. As we will see later in Sec.(2.6), these algebras are already in a cyclic form. So we do not make a crucial restriction when we demand cyclicity but only make some assumptions on the form of the algebra.

2.3 Hopf*-algebras

In the previous section, we saw that a linearisation of the unit, inverse and multiplication of a group yields a *-algebra. However, there is something essential missing: We also have some underlying set of group elements. Because of this underlying set, we can do a bunch of trivial things: First, we can take some element and simply forget it. This operation corresponds to the existence of a function $G \to 1$ from the group to some trivial one-element set. Second, we can check if two elements are the same. This corresponds to a function $G \to G$ that takes an element and maps it to itself. Third, we can copy group elements.

This corresponds to a function $G \to G \times G$ that takes an element and maps it to two copies. If we write these three objects in a linearised language as above, we get the 1, 2 and 3-leg tensors of the following set of tensors:

(8)

Remark 5. As in Eq.(1), the indices are labeled by elements of G and are identified in counter-clockwise sense with the labels a, b, c, d, ... Again, they have a very similar structure (delta functions with different numbers of indices), so we again added 0 and more-leg tensors with the same structure. Also we used the same symbol (circle with black content) to denote the different tensors and only distinguish them by the number of legs. Note that we will also use a smaller full black circle for general n-leg delta-functions over arbitrary spaces in some specified basis, whereas the bigger circle is reserved for the context of Hopf*-algebras.

Remark 6. These new kinds of tensors are also essential for the notion of symmetry: Imagine that we have some group $\mathcal{G}$ that acts on some set A via some permutation representation $X_g : A \to A$ and on some set B via $Y_g : B \to B$, and possibly also on other sets. Mathematical objects of interest are usually elements of more complex composed sets, like tuples $A \times B$, functions A^A, subsets 2^B, and so on. It is thus very natural and fundamental to ask how such objects transform under the symmetry. Let's consider as an example a function $c \in A^B$ from B to A. We would like that

$$c(b) = a \Leftrightarrow c'(b') = a',$$

so we immediately know that c transforms like

$$c'(b) = X_g(c(Y_{g^{-1}}(b)).$$

So in order to transform the composite object c, we have to simultaneously act with the permutation representation on every (incoming and outgoing) slot of c, each time with the same group element g (apart from inversion for the incoming slots). So in order to be able to apply the symmetry to a composite object, we have to be able to copy group elements for which we need the 3-leg tensor defined above. Also the 1-leg tensor is important: It tells us what it actually means that an object is symmetric (or better, *invariant* under the symmetry). To this end, the object has to stay the same (factor 1) for every single group element.

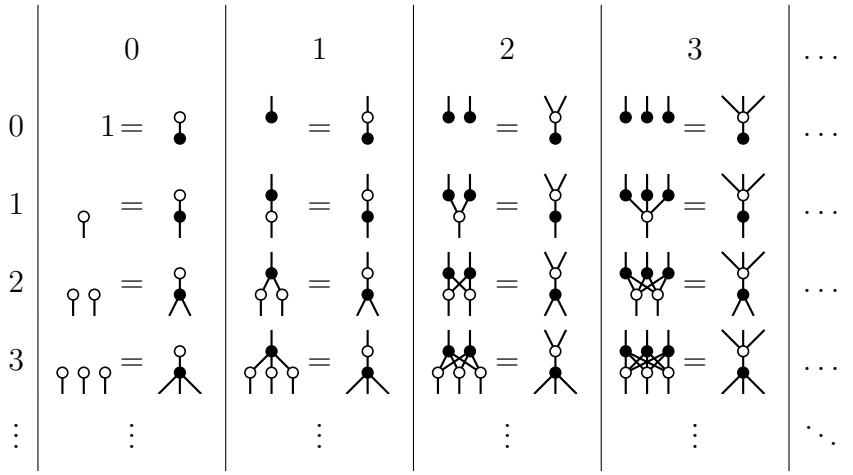
If we go over to a *-algebra as described in Sec.(2.2), it is not clear any more, what it means to "copy" an element of the algebra, or that something stays the same "for the whole algebra". That's why we also have to include the tensors Eq.(8) into our generalized notion of symmetry. For this we have to find axioms that describe those tensors.

Proposition 2. The tensors Eq.(8) also fulfill the axioms for a *-algebra Eqn.(2),(3),(4),(5).

Remark 7. As those delta functions are very special, it is easy to find a lot of additional axioms, for example commutativity. However, we do not require those here, as this leads to a nice duality between Eq.(8) and Eq.(1).

Remark 8. If we regard Eq.(8) as C*-algebra, the 2-leg tensor viewed as a map from one index to the other is not the identity any more but involves complex conjugation.

Apart from the *-algebra axioms for both Eq.(8) and Eq.(1), there is an important connection between them: As illustrated in Rem.(6), the tensors Eq.(8) define how we trivially (1-leg), conjugately (2-leg) and simultaneously (3-leg) act on different slots. If we impose that those deduced actions are again actual representations of the group, we end up with exactly the following additional axioms connecting the dual and direct part:



(9)

Now we have all structures and axioms that characterize symmetry in the discrete setting, and we can generalise them to the linearised setting:

Definition 2. We will call a *Hopf*-algebra* two sets of tensors with 0, 1, 3, 3, ... legs that fulfill each the axioms Eqn.(2),(3),(4),(5), and together fulfill the axioms Eq.(9). Each set defines a *-algebra, that we will call the *direct* and the *dual* algebra, respectively. We will adopt the convention that the contents of the direct and dual tensors are negatives to each other (as e.g. black and white).

Remark 9. In practice, we always have *Hopf*-algebras* where the dual and direct algebras are both C*-algebras. This is possible as the complex numbers Eq.(7) form a Hopf*-algebra when taken as dual and direct algebra at the same time.

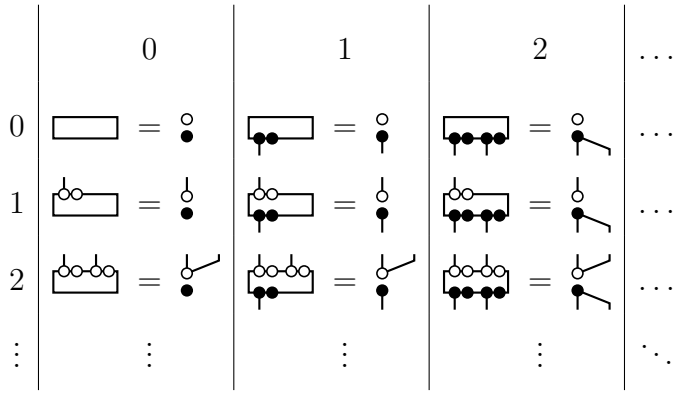
Remark 10. In finite dimensions, our definition seems to be equivalent to what is known as *Hopf C*-algebra* or *involutive Hopf algebra*. Though it is

quite different from the conventional language: Hopf C*-algebras are usually defined as (more general) Hopf algebras as introduced in Sec.(2.3), equipped with an additional *-structure on only the direct part with certain compatibility conditions. Also, we impose a cyclic form by Eq.(2) that is usually not required.

By construction, every group defines a *group Hopf*-algebra* with the direct and dual tensors defined in Eqn.(1),(8). But interestingly, there is a rich variety other possibilities. The dual symbols for every group Hopf*-algebra are commutative, but we did not not include this in our axioms. Because of this, they are completely symmetric under exchange of direct and dual tensors. So one can simply exchange the direct and dual tensors of a group Hopf*-algebra (in general of any Hopf*-algebra) to obtain a first new example, the *dual of* a group Hopf*-algebra. Another way to constructively get new examples of Hopf*-algebras is via the so-called bi-crossed product, which will be discussed in Sec.(2.8). But also apart from these constructions there are standalone examples, e.g. the smallest non-group Hopf*-algebra is of dimension 8 [2].

2.4 Useful equations

In this section, we will use the axioms of a Hopf*-algebra to derive other useful equations and expressions that we might use later.
First of all, the following set of equations follow from application of the fusion axioms and Eq.(9:2,2) (for some of them we also need the involution axiom Eq.(5):



(10)

Then, we can generalise conjugation from group algebras to general Hopf*-algebras. The following tensor network conjugates some algebra element (in and outgoing indices from bottom to top) with another one (index on the left):

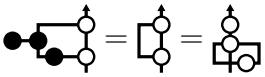
(11)

We will call this *direct conjugation* and also define dual conjugation as Eq.(11) with the dual and direct tensors reversed.

Example 2.1: For a group algebra, this is nothing but the usual conjugation: Multiplication from the left, inversion, multiplication from the left with a copy of the element, again inversion yields $g \to g^{-1}x^{-1} \to xgx^{-1}$.

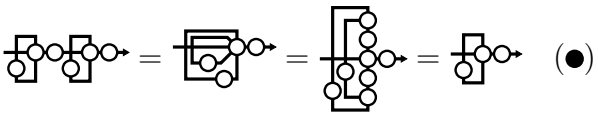
Example 2.2: If instead the algebra is the dual of a group algebra, the conjugation is quite trivial: it is the identity operator if the conjugating element x is the identity and 0 otherwise.

The direct (dual) center of an algebra is the subspace of elements that commute with all other elements under direct (dual) multiplication. We can project on this direct (dual) center (up to a factor) by conjugating with all elements in the case of a group algebra. For general Hopf*-algebras, "all elements" is defined by the dual (direct) unit, so a projection onto the direct (dual, after swapping direct and dual tensors) center (apart from some factor) is:



(12)

The correct normalization factor is just the inverse of the dual (direct) 0-leg tensor:

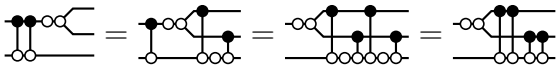


(13)

There is also an operation on two algebra indices that we will call *mutual conjugation* and that will become important later in the context of the quantum double:

(14)

For this mutual conjugation we have a similar pull-through condition as Eq.(9:2,2):



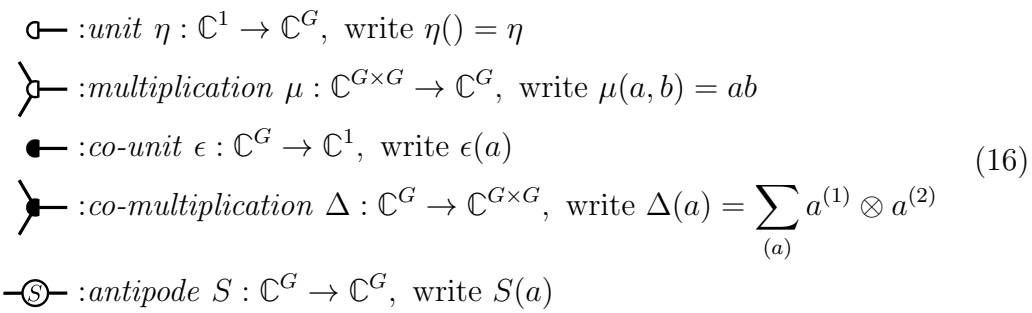
(15)

2.5 Connection to conventional Hopf algebras

In Sec.(2.3), we chose a set of structures and corresponding axioms that define a group, and then looked for arbitrary tensors fulfilling these axioms. Of course, the structures and axioms we used are not the only possibility to do so. In the mathematical literature, the linearisation (or *quantization*) of a group is usually taken to be a more general algebraic structure, namely a general *Hopf algebra*. In this section we will shortly explain the relation and differences between the latter and our Hopf*-algebras from Def.(2).

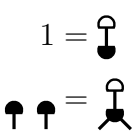
A general Hopf algebra also consists of two sets of tensors, referred to as the *algebra* and the *co-algebra*. But every set contains only a 1-leg tensor (called *unit* or *co-unit*, respectively), and a 3-leg tensor (called *multiplication* or *co-multiplication*, respectively). Instead of the direct and dual 2-leg symbols one

introduces a common 2-leg tensor S for both the algebra and co-algebra, called the antipode:

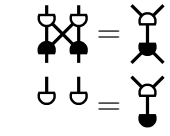


(16)

Instead of being a *-algebra, the algebra (co-algebra) is a unital associative algebra (co-algebra). A co-algebra is (in finite dimensions) equivalent to a (unital associative) algebra, just that the co-multiplication and the co-unit tensors are interpreted as a linear map in the reversed direction. Instead of the axioms Eq.(9), we have the following additional axioms for algebra and co-algebra together:



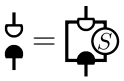
(17)



(19)

(18)

(20)



(21)

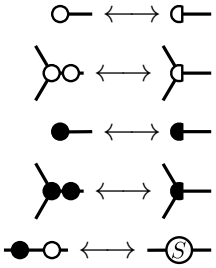
Let's summarize this in the following

Definition 3. A *Hopf algebra* is given by two pairs of a 1-leg and 3-leg tensor together with an 2-leg tensor S, as in Eq.(22), where both pairs form a unital, associative algebra (co-algebra), and all tensors together fulfill the axioms Eq.(17)-(20),(21). Without the axioms Eq.(21), the structure is called a *bi-algebra*.

Remark 11. Eq.(21) is the direct translation of the group axiom

$$gg^{-1} = 1 \text{ for all } g.$$

Proposition 3. Every Hopf*-algebra is also a Hopf algebra, with the following identification of the Hopf*-algebra tensors (left) with the Hopf algebra tensors (right):



(22)

The first axiom is directly equivalent to Eq.(9:0,0). The second and third ones follow from Eq.(9:1,2), (9:2,1). The fourth one corresponds to Eq.(9:2,2). The axiom Eq.(21) is equivalent to Eq.(10:1,1).

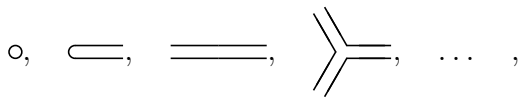
Remark 12. The converse of Prop.(3) is not true: there are Hopf algebras that cannot be turned into a Hopf*-algebra. It turns out that for finite Hopf*-algebras, this is possible if and only if S is an involution [2] (which is always the case for Hopf*-algebras).

So we have seen that our construction from Sec.(2.2) and Sec.(2.3) is not unique. However, we have the feeling that a Hopf*-algebra is the most natural mathematical object to describe symmetries in physical systems like the quantum double models. First of all, the corresponding structures and axioms Eq.(9) are much more symmetric and ordered. Second, the representation theory of C*-algebras is extremely beautiful and simple, whereas this is not the case for general unital algebras coming from general Hopf algebras. And third, the *-structure seems to be important to decompose the physical state space into symmetry sectors: only with this notion it is possible to define an inner product in the symmetry space and with that the isometry of representations.

2.6 Isomorphisms and the Fourier transform

So far we have been defining (Hopf)*-algebras via axioms, but we haven't talked about how different possible *-algebras can be characterized. It is well known that C*-algebras have a very simple classification which will be the subject of this section.
To this end, we first introduce another example of a C*-algebra (apart from group algebras):

Proposition 4. The *algebra of full N-dimensional matrices* is a *-algebra with the following 0, 1, 2, 3, ...-leg tensors:

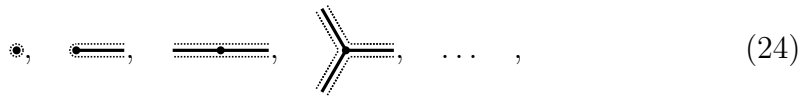


(23)

where the elements are labeled by a pair of indices corresponding to the row and column of the matrix entries. This N^2-dimensional *-algebra can be complexified to a C*-algebra with the two-dimensional algebra in Eq.(7). (One could have also attached one two-dimensional complex index to each real index, such that crossing them involves complex conjugation). Note that the order of the two indices is taken to be always the same relative to the direction pointing out of the tensor. So the 2-leg symbol viewed as map from one double-index to the other is not the identity but actually a hermitian conjugation. Similarly, the 3-leg symbol viewed as map from two of the double-indices to the third one is matrix-multiplication followed by hermitian conjugation.

We can now the matrix algebra Eq.(23) together with the delta algebra Eq.(8) to obtain another *-algebra:

Proposition 5. The *direct sum of matrix algebras* is a *-algebra with the following tensors:



(24)

where the algebra space is labeled by triple-indices. Note that here we used the slight generalization mentioned in Sec.(2.1): The dimension of the two dashed indices may depend on the value of the middle index, and is just the dimension of the respective direct sum component. The values of the middle leg of the triple index are then called *irreducible representations* or short *irreps* of the algebra. We will call the dashed two indices *right* and *left internal indices*.

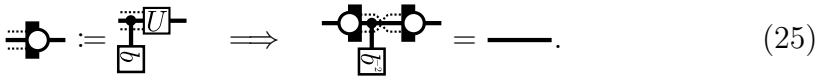
There are 3 kinds of operations one can apply to an algebra in the form of Eq.(24), that let invariant the axioms Eqn.(2),(3),(4),(5):

1. Applying the same unitary U to each of the (triple) indices, such that they cancel at each contraction. This is in fact a possible transformation for any algebra in any form.

2. Applying, for an arbitrary fixed irrep α, the same invertible matrix S_α to the left and its inverse S_α^{-1} to the right leg of each triple index doesn't change the algebra tensor at all.

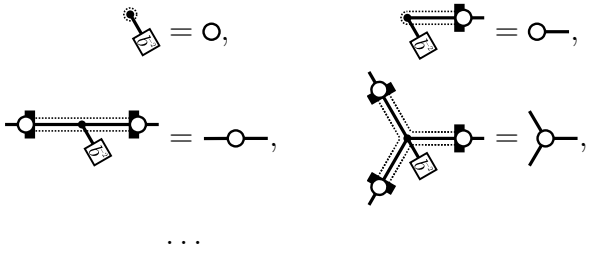
3. Multiplying each tensor by a factor b_α^{n-2}, where b_α is some number depending on the irrep α, and n is the number of legs of the tensor. Then the result of any tree-like contraction of algebra tensors is multiplied by some factor only depending on the number of open indices. This can also be done for any algebra in general, but only for the direct sum algebra, the factor b can depend on the irrep α.

The representation theory of C*-algebras tells us that any C*-algebra is isomorphic a direct sum of full (irreducible) matrix algebras as in Eq.(24). In our setting, this translates to saying that any C*-algebra (given in form of the corresponding tensors Eq.(1)) can be obtained from an algebra of direct-sum form Eq.(24) by applying first a a normalization with a factor b_α as in item 3 and then a unitary U as in item 1. We will call the product of the unitary U with the normalization by b_α the *diagonalising transformation*. We will draw this diagonalising transformation as a black bar half behind the tensor symbol of the diagonalised algebra:



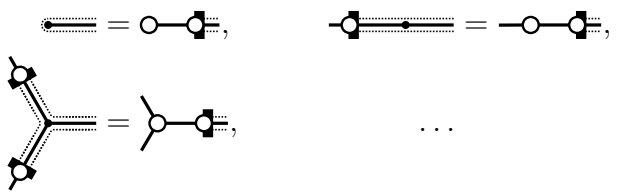
(25)

Applying the diagonalising transformation to every triple-leg of all n-leg tensors transforms the direct-sum algebra to the given form, apart from some factor of b_α^{-2}:



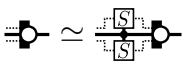
(26)

Alternatively, we can express these equation in some "pulling-through"-form, where we apply the inverse of diagonalising transformation to one leg in the equation above. This makes the normalization factor disappear:



(27)

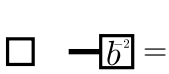
The diagonalising transformation is only determined up to some irrep-dependent unitary gauge on both indices:



(28)

We will speak (and have already spoken) of different sets of tensors arising from the same direct sum algebra Eq.(24) by such a transformation as the same algebra, but in different form.

Remark 13. For any group algebra, the diagonalising transformation Eq.(25) to the standard form Eq.(1) is known as the *Fourier transform*. The normalization factor b_α for the Fourier transform is



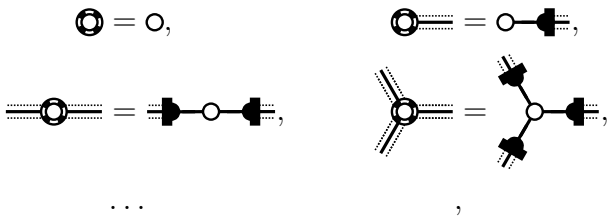
$$\text{or} \quad b_\alpha^{-1} = \sqrt{\frac{\dim(\alpha)}{|G|}}, \tag{29}$$

where the small circle around the irrep delta-tensor is a loop of an internal index and thus yields a value of $\dim(\alpha)$.

Remark 14. The dual algebra for a group Hopf*-algebra Eq.(8) is already in direct sum form: the index in the standard form corresponds to the irrep index, such that one just needs to add two dummy-indices of dimension 1 as internal indices.

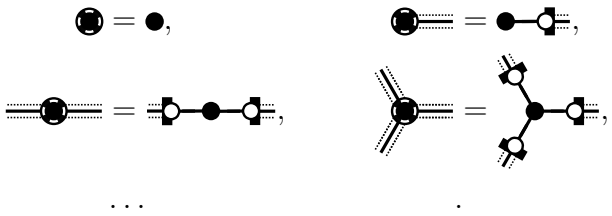
Now consider a Hopf*-algebra, and transform both the direct and dual algebra together by the operations defined above. The axioms Eq.(9) are still fulfilled after the transformation, iff the unitary U is the same for direct and

dual algebra, and the normalization factors b_α for the direct and dual algebra are inverses of each other. Then the transformations cancel at each contraction between direct and dual symbols in Eq.(9). Now we can bring the dual (direct) algebra into direct sum form (apart from the factor of b_α^{-2}) via the inverse diagonalising transformation, and transform the direct (dual) algebra accordingly with the inverse inverse diagonalising transformation. This will bring the Hopf*-algebra into some standard form with the dual (direct) algebra in direct-sum form. We will denote the direct algebra tensors in this form by the following symbols:



(30)

and the dual ones by:



(31)

For a group algebra, the dual algebra is already in direct-sum form. Thus, if we define the *generalized Fourier transform* to be the map which switches between the two forms, this definition matches the Fourier transform for group algebras. This Fourier transform can be written as:

(32)

2.7 Representations

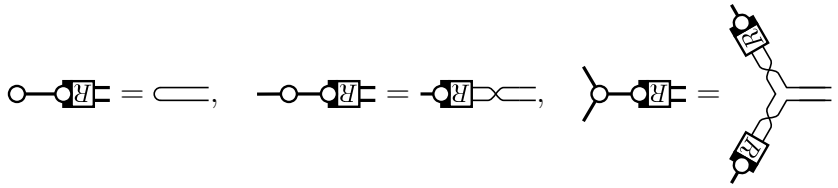
A direct (dual) representation of some Hopf*-algebra is nothing but a representation of its direct (dual) *-algebra. The tensor corresponding to a representation has one leg (the one at the tip on the right side) in the algebra vector space and two legs (the other two on the left side) in some representation vector space:

(33)

The corresponding shape is the symbol for the diagonalising transformation Eq.(25) of the direct (dual) algebra, combined with a rectangle that contains the label of the representation. The two legs on the left together index an operator on the representation vector space. The whole tensor can be interpreted as a map from the algebra to the operators of the representation vector space.

The tensor defines a representation if (interpreted as such a map) it is a homomorphism between the direct (dual) algebra and the operator algebra over the representation space.

Definition 4. A tensor as in Eq.(33) is called *direct representation* if it fulfills the following axioms:



(34)

A *dual representation* is defined analogously.

Example 2.3: The *right-regular* and *left-regular* representations are given by:

(35)

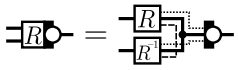
The defining properties Eq.(34) follow directly from the axioms Eq.(2), (3), (4).

Remark 15. Although the dual (direct) algebra is not involved in the definition of a representation, it is closely connected to the notion of representations: The dual (direct) 1-leg symbol corresponds to the trivial representation (which acts on a 1-dimensional representation space, thus those legs can be omitted), the dual (direct) 2-leg symbol defines the conjugate representation, and the dual (direct) 3-leg (and higher) symbols allow to build (conjugate) tensor products of representations (i.e. the simultaneous action of the same algebra through different representations on the tensor product space of the single representation vector spaces):

(36)

It is precisely the axioms Eq.(9) that guarantee that the trivial representation, the conjugate of a representation, and the tensor product of representations are again a representation.

The representations of a C*-algebra are as easy to classify as the algebras themselves: every representation is isomorphic to a direct sum of different irreducible representations with different multiplicities. If the algebra is written in direct-sum form, an irreducible representation is nothing but the map from the algebra space to one single irrep block. Thus all representations of the algebra in direct-sum form can be written as:

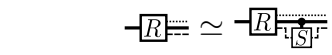


(37)

for some invertible operator R. Note that the dimension of the dashed index can depend arbitrarily on the irrep value, independently of the dotted internal indices that have dimension $\dim(\alpha)$. Those dimensions are called *multiplicities* of the different irreps. E.g. if the representation is some irreducible representation, all multiplicities are 0 apart from the chosen irrep value where it is 1. The operator R can be seen as a change of basis to coordinates labeled by irrep α, multiplicity m and internal state x.

Definition 5. If the operator R can be chosen unitary, such that we can replace R^{-1} above with R, we will call the representation *isometric*. So for isometric representations this new basis is orthogonal.

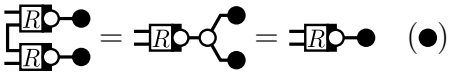
Remark 16. The operator R is only defined up to some irrep-dependent invertible gauge S_α on the multiplicity index:



(38)

> **Example 2.4:** The right- and left-regular representations from Ex.(2.3) are isometric representations. The operator R is nothing but the operator U from Eq.(25), so the diagonalising transformation apart from some factor.

The dual 1-leg tensor is the trivial representation, so we can build the projector onto the symmetric subspace of any representation by contracting with this dual 1-leg tensor, apart from some factor. This factor is nothing but the inverse of the dual 0-leg tensor, as we have:



(39)

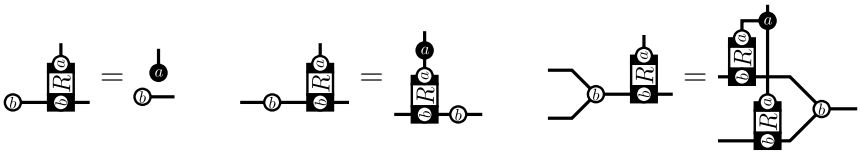
As the trivial irrep is always one-dimensional, its multiplicity is nothing but the trace of the projector resulting from Eq.(39). In general, we can construct the projectors onto arbitrary irrep sectors by contracting the representation with the central idempotent of that irrep.

2.8 Bi-crossed products and the quantum double construction

One way to constructively generate new examples of Hopf*-algebras is the *bi-crossed product* [14]. This is a simple generalization of the concept of bi-crossed product for groups which is in turn a generalization of the well-known semi-direct product of groups. For a bi-crossed product, not only one group acts on the other via some subgroup of automorphisms, but at the same time vice versa. In the context of Hopf*-algebra, this action via automorphisms becomes an *automorphism representation* of one algebra on another. By this we mean a representation as defined in Eq.(34) of a Hopf*-algebra a on the space of another Hopf*-algebra b that we denote by the following symbol:

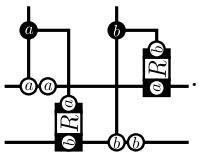
. (40)

This tensor has to be "an automorphism for each algebra element" what in the language of Hopf*-algebras translates to:



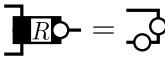
(41)

For the bi-crossed product of to algebras a and b, one needs two such automorphism representations (one of a on b and one of b on a) that have to obey a property that is called *matched*. The bi-crossed product of a and b is then an algebra that lives on the same space than the tensor product of the two algebras. This tensor product of a and b is a special case of the bi-crossed product with both automorphism representations trivial. Also for general bi-crossed products, the dual algebra of the product is always the tensor product of the two dual algebras of a and b. The interaction via the automorphism representations is implemented in the direct part of the product algebra. One can define this direct part by a representation that is isomorphic to the left-regular one:



(42)

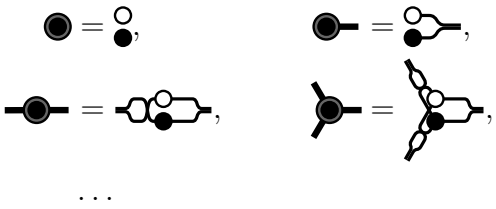
The upper two legs correspond to the algebra element that is represented, and the representing operator (from left to right) acts on a space that is a tensor product of the algebra space of a (top index) and b (bottom index). The representation obviously splits into two parts. The first part is a representation of a, namely the tensor product of the left-regular (on the upper index) and the automorphism representation on the space of b (on the lower index). The second part is just the reversed thing. Only because of the interaction via the two automorphism representations, this is not a representation of the tensor product of a and b, but of some other algebra, namely the bi-crossed one.
The *quantum double* (or *Drinfeld double* [13]) is a particularly well-known special case of a bi-crossed product. This is partly because of a property called *quasi-triangularity* that is important for the possibility of braiding statistics, but that we will not examine further. The two algebras a and b involved in the quantum double $D(\mathcal{G})$ are a freely choosable Hopf*-algebra $\mathcal{G}$ (that we will again label by white and black content) and its dual. Both automorphism representations are the conjugate left-regular representations acting on the space of the dual algebra identified with the one of the non-dual algebra. Here the representation of the direct on the dual part, the other one is completely analogous:



(43)

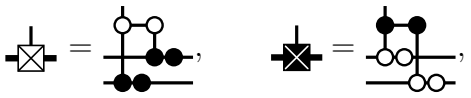
Those are automorphism representations as defined in Eq.(40) precisely because of the Hopf*-axioms Eq.(9:2,x) or Eq.(9:x,2). As a bi-crossed product, the dual

tensors of the quantum double are nothing but tensor products of the dual and direct (dual of dual) part of the original algebra:



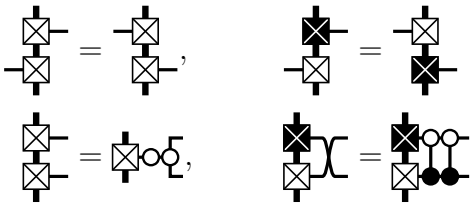
(44)

For the direct part of the quantum double, we define the following shortcuts for the two parts of the representation in Eq.(42):



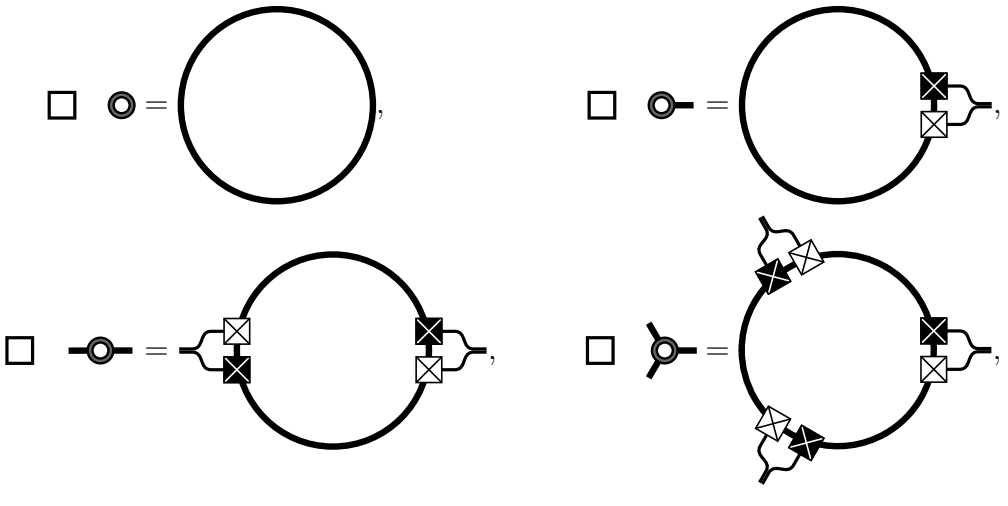
(45)

for which we find the following rules:



(46)

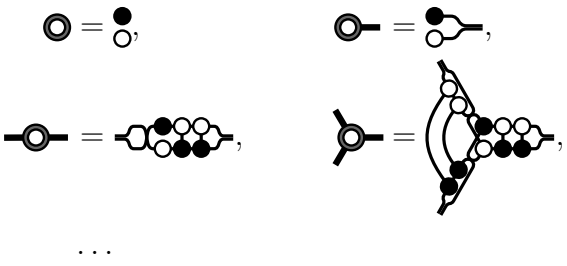
With this abbreviated notation, we define the symbols of the quantum double by tracing over products of the representation Eq.(42):



(47)

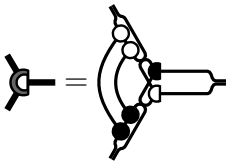
This form of the algebra tensors is manifestly cyclic, and symmetric with respect to the original direct and dual part. If we do not insist on this explicit cyclicity,

we can simplify the 0, 1, 2 and 3-leg tensors individually using the rules Eq.46:



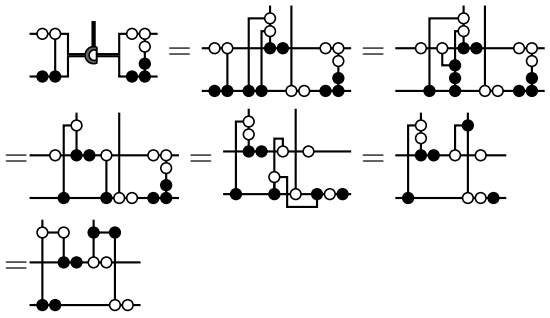
(48)

In particular, the left-regular representation (3-leg symbol combined with 2-leg symbol) looks like the following:



(49)

So it is almost the same as for the direct product of the two algebras, just that the first part of one incoming double index and the second part of the other incoming double index are mutually conjugated.
The product of the full and empty tensor in Eq.(45), which is nothing than Eq.(42) for the quantum double is isomorphic to this left-regular representation shown in Eq.(49):



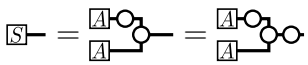
(50)

As a bi-crossed product, the quantum double algebra naturally contains both the direct and dual algebra as sub-algebras. This direct (dual) sub-algebra is obtained by closing the other part of the double-index by the dual (direct) 1-leg tensor.

3 Local symmetries and excitations

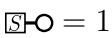
3.1 Local (quantum) statistical systems

In the following, we consider classical statistical or quantum mechanical systems. Observables as well states in these systems are given by elements of some vector space, which has the additional structure of a C*-algebra. For classical systems, this is just the delta-algebra Eq.(8), whereas in quantum mechanics, one has a full matrix algebra Eq.(23). The 2-leg tensor of this algebra defines an inner product of the space of states and observables. States of the system can in fact not be arbitrary, but have some positivity constraint. Every valid state S has to be of the following form:



(51)

for some vector A. As the equation shows, states are automatically self-adjoint by axiom Eq.(3). They are thus restricted to some real sub-vector-space of half of the (real) dimension of the full complex vector space. As having two independent copies of the same element A is something non-linear, the set of valid states is not a linear subspace. If one further normalizes the state with respect to the overlap with the 1-leg tensor:



$$= 1 \tag{52}$$

the set of states is actually a convex subset.
Fundamental physical systems consist of a large set of degrees of freedom $f \in F$, each of which corresponds to one local system called *spin* with the C*-algebra of observables realized as an h_f-dimensional vector space. The total space of observables/vectors is then given by

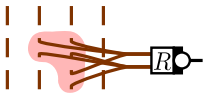
$$\mathcal{H} \coloneqq \bigotimes_{f \in F} H_f = \bigotimes_{f \in F} \mathbb{C}^{h_f}. \tag{53}$$

The physical laws then have a notion of locality: The degrees of freedom can be distributed over a lattice of finite dimension, such that the physical laws can be formulated as a set of restrictions on small patches of neighboring spins in that lattice. This is an extremely strong restriction on the physical laws for large systems, but without such an assumption it is impossible to find out something about the physical laws at all.

3.2 Local, commuting symmetries

Due to the locality of the physical laws it is natural that also symmetries of the physical laws are local. This means that one has a set of local symmetries defined on some patches (*sites* $s \in S$) of neighboring spins. Typically, these patches cover the whole lattice, and overlap with each other. As we are in a

probabilistic or quantum setting, each symmetry located at site s is given by some representation R_s of a Hopf*-algebra $\mathcal{G}_s$ acting on a few spins around s. So the global operator corresponding to one local symmetry R_s can look like:



(54)

Now we make two assumptions on the symmetry:

- All representations are isometric according to Def.(5).
- All representations commute among each other.

Remark 17. Especially the second requirement is very strong, and one will not find such a symmetry for generic systems. But it holds for a class of exactly solvable systems that are typically fixed points of some RG flow. Examples are the classical Ising models at zero or infinite temperature, all string-net models and the quantum double models that will be studied extensively later.

Since all symmetries commute, they act independently of each other, thus the product of all R_s forms a representation of the global tensor product algebra $\mathcal{G} = \bigotimes_{s\in S} \mathcal{G}_s$. This representation can be used to decompose the physical state space into irreducible spaces (*superselection sectors*) labeled by an irrep α of $\mathcal{G}$ and a multiplicity index m:

$$\mathcal{H} = \bigoplus_{\alpha,m} V^{\alpha,m} \quad \text{, where } \alpha = (\alpha_1, \ldots, \alpha_S), \text{ and } m = 1, \ldots, M_\alpha \tag{55}$$

As $\mathcal{G}$ is a tensor product algebra, its irreps are given by lists of irreps of the $\mathcal{G}_s$.

Remark 18. The subspaces are orthogonal because of the first assumption of isometric representations. The decomposition of the full irrep sector labeled by α into multiplicity subspaces indexed by m is only unique up to an arbitrary unitary gauge on m.

One is mostly interested in subspaces with most of the α_s equal to the trivial irrep. If α_s is non-trivial for some s, one speaks of the corresponding states in the superselection sector as having an *excitation* or *particle* at s. States within the subspace for all irrep components equal to the trivial irrep are called *ground states*.
We can now further decompose those invariant subspaces by choosing an orthonormal basis within each of them, labeled by $x = 1, \ldots, \dim(\alpha)$:

$$V^{\alpha,m} = \mathrm{Span}\{V_x^{\alpha,m}\}, \text{ where } x = (x_1, \ldots, x_S) \text{ and } x_s = 1, \ldots, \dim(\alpha_s) \tag{56}$$

As the irrep α is just a tuple of irreps for the single components, its dimension is nothing but the product of the dimension of all these components. Thus we can also split the label x into local components. Apart from this notion

of locality, the x label is chosen completely arbitrary and defined only up to a local unitary gauge for each single component. So what we get in total is a change of basis from the local computational spin basis to the one labeled by α, m, and x:

$$\begin{aligned} & |i_1;\ldots;i_F\rangle\,, && \text{with } i_f = 1,\ldots,h_f \\ \longrightarrow\ & |\alpha_1,x_1;\ldots;\alpha_S,x_S,\quad m\rangle\,, && \text{with } \alpha_s \text{ irrep of } \mathcal{G}_s, \\ & && x_s = 1,\ldots,\dim\alpha_s,\quad m = 1,\ldots,M_\alpha \end{aligned} \tag{57}$$

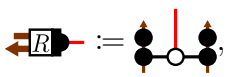
Whereas the α_s and x_s label can be associated with some site s, the m label has a priori no notion of locality. The x label is indeed a completely local degree of freedom, as every operator on the space spanned by

$$\mathrm{Span}_{x_s}\{|\alpha_1,x_1;\ldots,\alpha_S,x_S,\quad m\rangle\}$$

for all other α and x and the m label fixed can be implemented by an operator that is local around s: As this space is irreducible under the action of the algebra component $\mathcal{G}_s$, those operators are nothing but the representing operators of R_s which are local by construction. Also the α_s-label can in principle be changed by local operators, but in general changing the α component at one site s will automatically also change some of the α-labels around s. The m label can not in general be split up into localized parts, but if the symmetry is not restrictive enough, it can contain parts that can be changed by local operators. For example if some spin is not involved in any symmetry representation, it can be seen as component of m.

Example 3.1: As example, let's consider the 1-dimensional classical infinite-temperature Ising model. The symmetry is given by Z_2, acting independently on each single spin by a regular representation. Each irrep component is either the trivial irrep $(1,1)$ or the non-trivial irrep $(1,-1)$. As Z_2 is commutative, the x-label is trivial (one-dimensional for both irreps). As additionally the symmetry representations do not overlap, the m-label is also trivial ($M_\alpha = 1$ for all α).

Example 3.2: As a second, slightly more non-trivial example, take the one-dimensional classical zero-temperature Ising model without fields on a ring (periodic boundary conditions). Here, the symmetry algebra is the dual of Z_2 (which is in fact isomorphic to Z_2 itself). It acts locally on pairs of neighboring spins and projects onto the subspaces where their values are either the same or different:



(58)

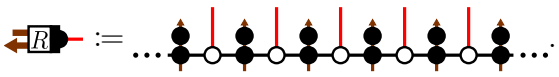
where the empty and full tensors are those of the group algebra of Z_2. The x-label is again trivial, but the m-label is not: For α where all the

components fuse to the trivial charge (that is, the number of non-trivial excitations is even) $M_\alpha = 2$, otherwise $M_\alpha = 0$. The reason is simply that the number of domain walls on a ring always has to be even, and for any valid domain wall configuration there are two possible states which are connected by a flip of all spins.

3.3 Tensor network representations

The representation of $\mathcal{G}$ on the global state space can easily be written as a tensor network operator. To this end we simply multiply the single representation operators Eq.(54) at the physical level. This leads to a tensor network operator with the same locality structure than the system, because only the multiplication of representations with overlapping patches is non-trivial and yields virtual indices. As the single representations commute among each other, the order of multiplication doesn't matter. From this global representation, we can build the projectors onto arbitrary irrep sectors and implement arbitrary operations on the internal state inside of each single irrep. If we want a state with a given excitation configuration, we can simply apply the projector onto the corresponding irrep sector (possibly with some non-trivial operation on the internal state) to some arbitrary state, and hope that the output is non-zero.

Example 3.3: Let us examine this for the zero-temperature Ising model in Ex.(3.2). After multiplication of all local representations, one obtains the following tensor network:

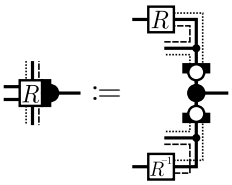


(59)

This is just the projector onto a domain-wall configuration, given by the red indices.

3.4 Creation and topology of excitations

Excitations can be created by local operations. This can by done by simply taking the symmetry representation in the form of Eq.(37) and replacing the direct tensor in direct-sum basis by the dual tensor in the same basis:

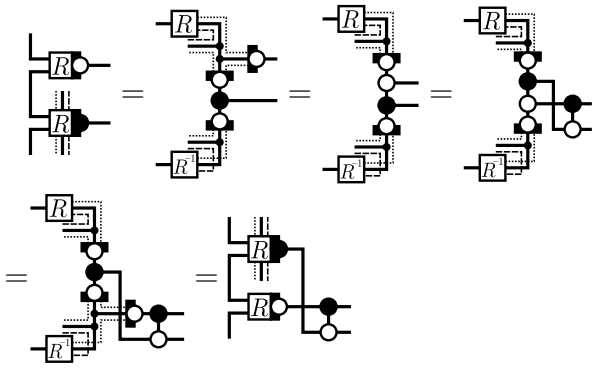


(60)

This is an operator (left pair of indices) on the representation space that creates a particle determined the algebra element on the right, or better, fuses an already existing excitation at this site with the new particle.

Remark 19. The additional indices on the top and bottom determine the multiple ways in which the particle can be created: The given operator might also create add additional particles nearby, and those indices can change where and how this happens. For example it might be the case that creation of a particle always creates a corresponding antiparticle, so the index determines whether this happens to the left or to the right. Though, we do not claim that we actually need all of the open legs in Eq.(60) to get the most general creation operator. Also, the creation operators should form a representation of the dual Hopf*-algebra, which is not the case for all possibilities to close the additional open indices.

It is easy to see that the excitation at the corresponding site gets fused with the one we put to the right open index of Eq.(60). To this end we look at the action of the symmetry after the application of Eq.(60):



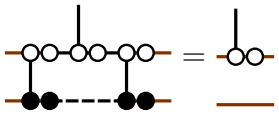
(61)

In step 1, we cancelled R and R^{-1} and used the commutativity of the small black delta-tensor on the irrep space. Then we used Eq.(27) to get the direct algebra tensor from its direct-sum form. In step 3, we used the Hopf*-axiom Eq.(9:2,2), then we reversed the previous steps at the bottom of the expression. The resulting equation tells us that the symmetry action after the creation of the particle corresponds to the original particle fused with a copy of the new particle.

Example 3.4: Let's stay for a moment with the simple zero-temperature Ising model. The R-operator has the following form:

(62)

Note that the internal state and the diagonalising transformation is trivial. So the operator creating a charge (when we trace over the two additional open multiplicity degrees of freedom) is:



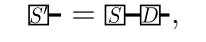
(63)

This is nothing but a flip of one of the pair of spins. If we would have chosen other values at the additional open legs in Eq.(60), we would have gotten other operators, like a flip of the second spin. All possible operators have the effect of creating a domain wall between the pair of spins.

We already mentioned that in general, excitations can only be created in pairs or triples and so on. One can then try to re-eliminate those accidentally created particles which leads to further particles around, and so on. Depending on the pattern of representations, this will lead to structures of excitations with different topology: If the patches for the different sites do not overlap, excitations are completely localized and can be created by local operators. If creating a particle will create its antiparticle at some site nearby, excitations come in pairs (so structures of dimension 0) and can be created by operators along strings connecting them (so structures of dimension 1). If excitations are locally created in some ring-like configuration (small triangle etc.), excitations come as closed strings (so structures of dimension 1), and can be created by operators along surfaces/membranes connecting them (so structures of dimension 2), and so on. For a system of dimension d, this goes up to $(d-1)$-dimensional excitations created by operators in the d-dimensional region in between.

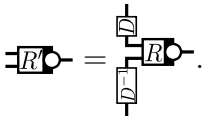
3.5 Deformation and symmetry breaking

As already mentioned, the requirements of commutativity and isometry of the local symmetries only hold for very special systems. So for generic states, the constructions above do not apply directly. Still one can hope that if the system is close to some RG fixed-point system, it behaves qualitatively similar. To this end, we slightly deform the system locally, e.g. by a tensor-network operator with the locality structure of the system and bounded bond dimension, from some RG fixed-point. If we have such an operator D mapping the ground state S of the fixed-point model to a new ground state S'



(64)

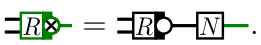
we can at the same time transform our symmetry representations from R to R':



(65)

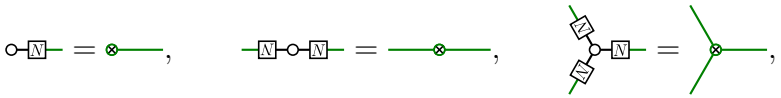
So we still have local, commuting symmetries. But R' not isometric any more if D is not unitary. Thus, the deformed excitations would be non-orthogonal, and the deformed Hamiltonian non-hermitian. It might still be possible to find commuting isometric representation of the same algebras, if one doesn't demand anymore that they are strictly local. Instead, one might hope, that they can be efficiently approximated by local representations (i.e. exponentially well in the size of their support). While this might be the case for small deformations, it will not be true anymore for at least some part of the symmetry, and thus a

phase transition will occur.
By deforming, we can at some point reach another RG fixed-point system, such that there has to be some transition along the deformation path between the qualitative behavior of the old and the new RG fixed point. Typically, one deforms from some fixed point to another one with a reduced symmetry. This phase transition to a new qualitative behavior is called *symmetry breaking*. By reduced symmetry we mean the original representations, but now restricted to a sub-Hopf*-algebra by some projector N:



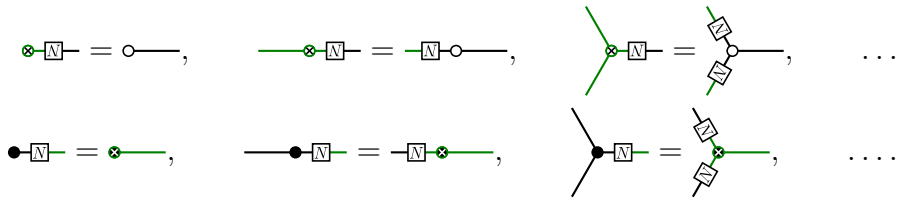
(66)

Actually, N is an isometry between the full (black/white) and the reduced (cross/anti-cross) algebra (for clarity, the reduced algebra lines and frames are colored green), such that we can identify the green algebra space with a subspace of the black one. Here we mean by sub-Hopf*-algebra that the restriction by N of the direct and dual tensors of the full algebra is again a Hopf*-algebra:



(67)

and the same for all dual tensors. Technically, the reduced representation Eq.(66) is not a representation anymore in the sense of Eq.(34). If this is our aim, the following definition (which includes the the original identity is contained in the sub-algebra) would be more natural:



(68)

The problem with the sub-algebra as in Eq.(67) is that the representation of the reduced unit doesn't have to be the full identity in general but can be the identity on a subspace only (and zero on the rest). So it might happen that some irrep subspaces of the full representation are completely mapped to 0 by the reduced representation. This phenomenon is called *confinement*. Of course, every invariant subspace with respect to the full symmetry algebra is also invariant with respect to the reduced symmetry. But they are not irreducible any more in general, such that they split up in different irreps of the sub-Hopf*-algebra. Thereby, different irreducible (under the sub-algebra) sub-sub-spaces emerging from different irreducible sub-spaces (under the full algebra) might belong to the same irrep. When this happens, one speaks of *condensation*.

Example 3.5: Consider the symmetry breaking from the group Z_4 (with elements 0, 1, 2, 3) to the subgroup Z_2 (formed by 0 and 2). As this subgroup is a sub-Hopf*-algebra in the sense of Eq.(68), there is no confinement. The 4 irreps of Z_4 (that we call 1, i, -1, $-i$ by their phases) split

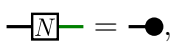
up into irreps of Z_2 (namely 1 and -1) in the following way:

$$\begin{aligned} 1 &\Rightarrow 1 \\ i &\Rightarrow -1 \\ -1 &\Rightarrow 1 \\ -i &\Rightarrow -1 \end{aligned} \tag{69}$$

so the irreps 1 and -1 and the irreps i and $-i$ are condensed together. There's no actual splitting up of irreps as we have abelian groups.

Example 3.6: As an example for confinement, consider the dual of some group algebra $\mathcal{G}$ that we break down to the dual group algebra on some subgroup $\mathcal{N}$ of $\mathcal{G}$. Although $\mathcal{N}$ is a sub-algebra of $\mathcal{G}$ in the sense of Eq.(68), this is not true for the duals. Irreps are given by single group elements. The irreps that are not contained in $\mathcal{N}$ are confined. This is because for any representation of the dual of $\mathcal{G}$ (e.g. the left-regular one), the corresponding irrep sectors mapped to zero under the representation restricted to the elements of $\mathcal{N}$.

Example 3.7: Consider the sub-algebra containing only the trivial particle:



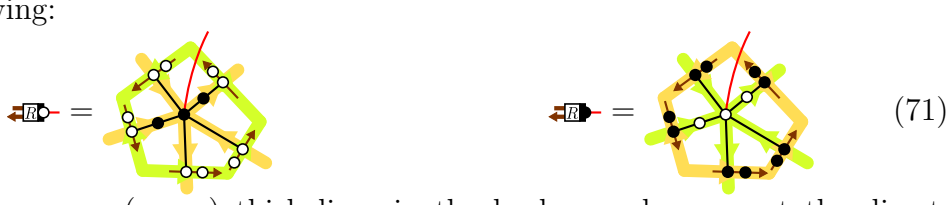
(70)

where the green index on the right is trivial (one-dimensional) and thus invisible. Under this restriction, all irreps except for the trivial one will be confined.

4 The quantum double models

4.1 Direct and dual symmetries in the quantum double model

A quantum double model [5] is specified by some finite-dimensional Hopf*-algebra $\mathcal{G}$ with basis set G and a two-dimensional (direct) lattice together with its dual lattice. We will call the vertices and edges of the direct and dual lattice *direct* and *dual vertices* and *edges*. Every pair of neighboring dual and direct vertex will be called a *site*. We also need an orientation attached to any direct and dual edge. The physical state space described is then given by density matrices of dimension G^E where E is the set of (direct or dual or pairs of direct and dual) edges of the lattice. We will imagine and draw the qu-G-it for a (direct or dual) edge as located at the intersection of the corresponding pair of dual and direct edges.
Then one defines symmetries given as local representations of some Hopf*-algebra(s) as described in Sec.(3). In our case, those representations are such that they always act as two independent copies on both the ket and bra layer. So our ground state will be a (zero-temperature) pure-state, and we can split the ket and bra layer completely and only work in the Hilbert space of pure states with one half of the symmetries. The symmetry algebras are both $\mathcal{G}$ and its dual, and we have one representation of $\mathcal{G}$ (the dual of $\mathcal{G}$) for each direct (dual) vertex. Each representation is nothing but the tensor product of (left- and right-) regular representations acting on the qu-G-its at the edges around the direct (dual) vertex. The exact form of the representation of the direct (left) or dual (right) algebra at a corresponding direct or dual vertex looks like the following:



(71)

Here, the orange (green) thick lines in the background represent the direct (dual) edges with their orientations indicated by arrows. The rest is the (direct or dual) representation written as tensor network. The brown legs at the edges correspond to incoming an outgoing physical indices for the respective qu-G-its, with the outgoing ones marked by arrows.

Remark 20. For non-commutative dual (direct) algebras, different positions of the red index with respect to the other indices of the dual (direct) symbol in the middle yield different direct (dual) representations. This position can be specified by the pair of spins in between that it is located, and this pair is shared with one unique neighboring dual (direct) vertex. By that, we can attach one direct and one dual representation to each neighboring pair of direct and dual vertex, i.e. to each site.

Remark 21. In the representation for the direct (dual) vertex, for each spin, one can have one or no dual (direct) involution on the connection with the central dual (direct) tensor, and the representation at the spin is either left- or right-regular. The first point depends on whether or not the dual (direct) edge of the spin is directed counter-clockwise around the direct (dual) vertex. The second point depends on whether or not the direct (dual) edge of the spin is directed towards or away from the direct (dual) vertex. If we add the direct (dual) 2-leg tensor to each the incoming and outgoing index of some spin in both the direct and dual representations, we get the direct and dual representations with the orientation of the corresponding direct (dual) edge reversed. So different orientations are not different models but rather different conventions that can be changed by redefining the corresponding spin. As everything following will be deduced from those representations, we can change the orientation of a direct (dual) edge for any tensor network that will occur by multiplying a direct (dual) 2-leg tensor two every physical (brown) leg corresponding to that edge.

Example 4.1: Consider the case of $\mathcal{G}$ being a group algebra. Then the direct representations are the so-called vertex operators A^g, and the dual representations are the plaquette operators B^h that act like the following:

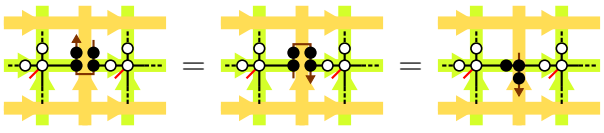
$$A^g \left|a, b, \ldots\right\rangle = \left|ga, bg^{-1}, \ldots\right\rangle \qquad B^h \left|a, b, \ldots\right\rangle = \delta_{ab^{-1}\ldots} \left|a, b, \ldots\right\rangle ,$$

where whether we have the left- or right-regular representation and whether we have the inverse or not in the product depends on the lattice orientations. For the group Z_2, the non-trivial operators A and B are the ones of the toric code:

$$A = \sigma_x^{\cdots} \qquad B = \frac{1 + \sigma_z^{\cdots}}{2}$$

Note that there's a subtlety when we also want to act with complex superpositions of group elements: Then, in the notation above, one would have to represent some spins as ket and some as bra.

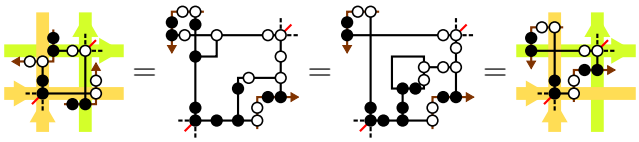
In order to define local excitations as in Sec.(3), one would like to have that all of those representations commute mutually. This is of course the case for all pairs of representations that do not have common spins they act on. But also pairs of direct-direct (dual-dual) representations with one common spin commute, as can be seen on the level of their actions on a single spin:



(72)

This can be seen using the flip rule Eq.(3:3) for the first equality together with the fusion rule Eq.(4:2,2) for the second equality.

Also pairs of neighboring direct-dual (dual-direct) representations that share two spins commute if the corresponding sites have no common direct/dual vertices, that is, if none of the red lines is positioned between the two spins shared by the the two representations:



(73)

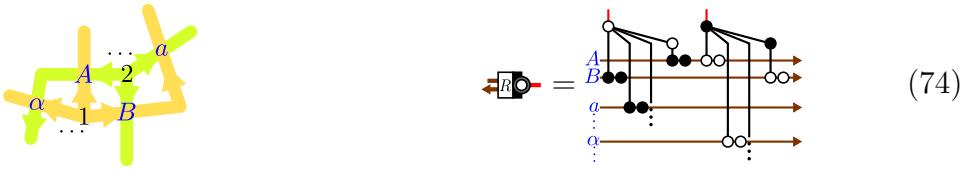
For the first equality, the Hopf axioms Eqn.(9:2,2), (9:2,1), (9:1,2) and (9:1,1) were used to interchange the actions of the dual and direct regular representations at each of the two spins. Also the order of indices was changed by the flip axioms Eq.(3:3). In the second step, the fusion axiom Eq.(4:2,2) was used three times for each the central dual and direct "copying" tensors. In the last step, Eq.(10:1,1) was used to get rid of the tensors in the middle.
In contrary, if at least one of the red indices of a dual-direct pair of representations is located in between the two shared spins, the representations do not commute in general. Also, two direct (dual) representations at the same direct (dual) vertex but with different positions of the red index obviously do not commute in general.

4.2 The quantum double algebra as symmetry

As we have seen, the local direct and dual representations do not commute mutually in general. This means that we cannot directly apply the construction from Sec.(3), because we cannot regard all representations together as representation of the direct product of all algebras. However, the direct (dual) representations of the dual (direct) center do commute among each other. This is because for elements of the dual (direct) center, we can move the red index where we want, so Eq.(73) always applies. This indicates that we can regard the product of a pair of neighboring direct and dual representations as a representation of a bi-crossed product of the direct and dual algebra.

Proposition 6. The product of a pair of neighboring dual and direct representations (with the red indices pointing towards each other, in between the shared spins A and B) is a representation of a bi-crossed product, namely the quantum double $D(\mathcal{G})$.

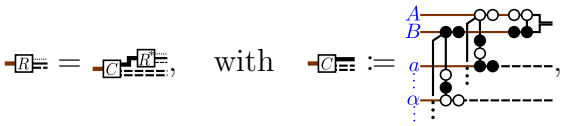
We will prove Prop.(6) by giving explicitly the tensor R from Eq.(37) for this representation. To keep things clear, we re-arranged the indices corresponding to all spins in the same horizontal direction. Without loss of generality (see Rem.(21) we can choose the edge orientations according to the lattice shown on the left. The product of the dual and direct representations corresponding to the site formed by the vertices 1 and 2 is shown on the right:



(74)

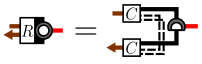
Remark 22. Whether the direct or the dual representation comes first is just a convention. We decided for the dual first as this matches the definition of the quantum double in Eq.(47).

Then the tensor R is given by:



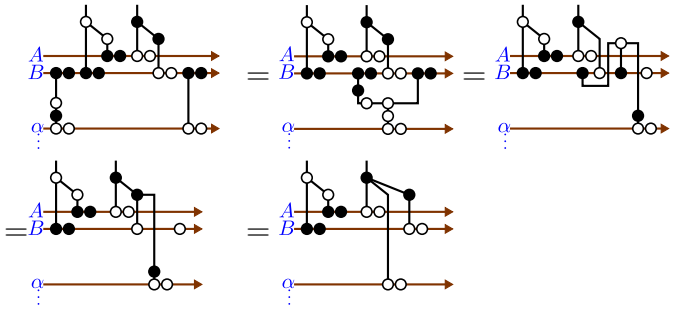
(75)

where R^* is the R-tensor of the regular representation (of the quantum double) as shown in Ex.(2.3). It only remains to show that R indeed generates the corresponding representation Eq.(74) as in Eq.(37). As we know that this is already true for R^* and the left-regular representation, Eq.(37) becomes:



(76)

To this end we split R into several parts each of which connects two spins. Those pairs are (from left to right) $\ldots - B$, $\beta - B$, $\alpha - B$, $\ldots - A$, $b - A$, $a - A$, $B - A$. The part $B - A$ and its inverse exactly turn the left-regular representation of the quantum double into the operator generally defined in Eq.(43) for the quantum double, as shown in Eq.(50). This operator is nothing but the representation Eq.(74) without additional spins apart from 1 and 2. The other parts connect the operator Eq.(43) on the spins A and B with the other spins, as shown here for the part $\alpha - B$, for other parts it is completely analogous:



(77)

Remark 23. From Eq.(76) we can see that the representation of $D(\mathcal{G})$ is isomorphic to a multiple of the left-regular representation with the multiplicity being $|G|$ to the power of the number of additional spins apart from A and B. Further, R^* as well as C are unitary, so the representation is isometric.

Remark 24. If both red indices were at different positions, the representations would commute, but it's not so clear what happens if only one red index points somewhere else. It is possible that they still form a representation of the quantum double or that they commute or even something in between. We would also get different (possibly more complex) algebras if we consider the product of more representations. Those would possibly together form representations of nested bi-crossed ("multi-crossed") products. In the end, we can regard the product of all direct and dual representations as the representation of some huge multi-crossed product with the "crossings" only locally between neighboring direct and dual representations.

4.3 Ground states and local excitations

When we construct the ground state projector from our local representations Eq.(71) by contracting them with the dual 1-leg tensor as in Eq.(39), the red index just disappears. The resulting projectors commute among each other as special case of Eq.(72) and Eq.(73). So we can define the ground state space as the common range of all those projectors. Now we would like build single excitations. If we build the projector onto one irrep subspace of a direct (dual) representation at some single direct (dual) vertex by contracting the representation with the corresponding algebra element, we find that this projector does not commute in general with the representation at the neighboring dual (direct) vertex to which the red index points to. To avoid this, we need to project our irrep to the dual center with Eq.(12) before we contract it with the representation.

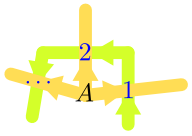
> **Example 4.2:** For $\mathcal{G}$ being a non-abelian group algebra, this is the case for the dual representations (plaquette operators). An irrep of the dual algebra is given by a single group element g. So one irrep projector would be the projector onto the subspace where the product around the plaquette equals g. But the ground state projector of the neighboring vertex between the first and last spin in the product conjugates the latter by all group elements:
>
> $$g \to \sum_h hgh^{-1}$$
>
> This averages g over the whole conjugacy class. So it makes only sense to consider class-function superpositions of group elements as single excitations, i.e. elements of the dual dual (thus direct) center.

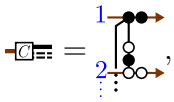
Now let's consider common excitations on a direct-dual pair of neighboring vertices. Those are given by projectors onto the irrep subspaces of the quantum double representation Eq.(74). As both red legs there are directed towards each other, those projectors commute with all ground state projectors, and all other double-representations that do not have intersecting direct or dual vertices. Note that all single excitations as described above are already contained in the

quantum double irreps.

4.4 String and ribbon operators

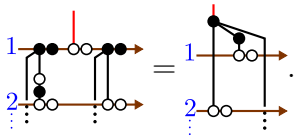
According to Eq.(60) from Sec.(3.4), we can build the operators that locally create excitations. For simplicity, we will first consider this for single direct (dual) excitations. Each single direct (dual) representation is isomorphic to a multiple of the regular one, and the operator C for this representation (analogous to the one for the quantum double in Eq.(75) is given by part 1 and 3 (2 and 3) of Eq.(75). It's shown here for a direct representation, for the dual one it's completely analogous:





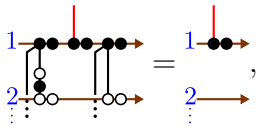
(78)

as we have



(79)

If we now construct the operator Eq.(60) with one possible choice of the additional open indices, we get the following creation operator:



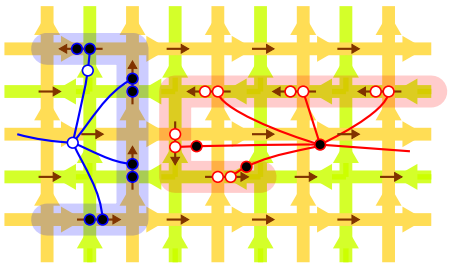
(80)

which is nothing but the dual (direct) regular representation on a single spin. Note that which spin this is depends on which spin we choose as the special one in Eq.(78), i.e. how we define our multiplicity degree of freedom. To create a single pure excitation, we have to contract the open red index with the corresponding irrep projector. Note again that, however, single direct (dual) excitations have to be in the dual (direct) center, thus it makes only sense to create the according superpositions of different groups of irreps.

> **Example 4.3:** For $\mathcal{G}$ being a group algebra, the direct creation operator is just the multiplication of a single spin with the corresponding character function. The dual creation operator is just a single spin flip (multiplication with some group element). However, for non-abelian groups, we cannot simply flip by a single group element, but we have to multiply with a whole conjugacy class (which is in the dual center of the dual of $\mathcal{G}$.

Obviously, the operator Eq.(80) is also a creation operator for the neighboring direct (dual) vertex that shares the selected spin. The particle it creates there is not the same, but the conjugate particle. The other direct (dual) local ground state projectors commute with the creation operator. This is also true

for the dual (direct) projectors, if the created particle is in the dual (direct) center. So in total, the operator Eq.(80) creates a particle-antiparticle pair on neighboring direct (dual) vertices. One could also say, the operator moves a particle from one direct (dual) vertex to the next one. We can now imagine to simultaneously move a particle from each vertex to its next neighbor in a chain of vertices. The obvious result will be an operator that moves the particle (or creates a particle-antiparticle pair) from one end of the chain to the other. Note that the simultaneous creation of charges is done via the dual of the dual (dual of the direct), thus the direct (dual) algebra. The corresponding operators are called direct (dual) *string operators*. The following graphics shows such a direct (dual) string operator in blue (red):



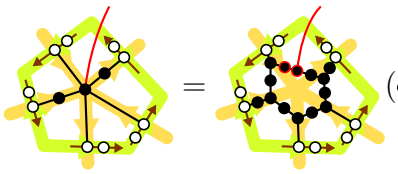
(81)

Interestingly, those operators have the same form as the dual (direct) representation itself, just that they act on arbitrary (possibly open) direct (dual) path in the lattice instead of the direct (dual) path around some dual (direct) vertex. Again, this operator only makes sense for elements of the dual (direct) center. We can do the same for the double representations Eq.(74). The resulting operators will move the double-excitations along some ribbon, i.e. a direct and dual path that are directly next to each other and always run in parallel. At each spin of the ribbon we have a representation of the dual algebra of the quantum double. This is either the regular representation of the dual or the direct part of this algebra, depending of whether the spin is at the inner or outer border of the ribbon. All of those representations are connected by the direct algebra of the quantum double. For group algebras, the explicit form of the ribbon operators is given in [4].
This one-dimensional form of the particle creation operators is one of the key features of topologically ordered systems. It is directly connected to the fact that the local creation operators described in Sec.(3.4) create exactly one other particle such that together they fuse to the trivial excitation. This is by far not the only possible behavior.

4.5 The quantum double symmetry as tensor network

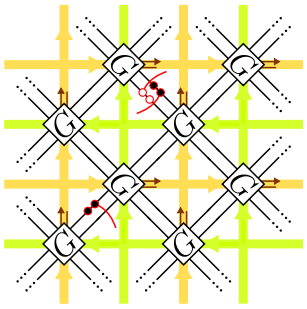
In this section we will create a tensor network that is a mixture of the ground state projector at most places and the representation of the local symmetries at some places. We begin with tearing the central dual (direct) tensor of the

direct (dual) representations apart:



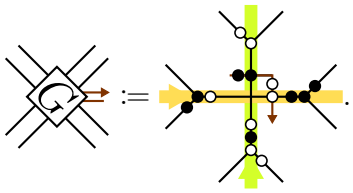
$$(\circ)^{-1}. \tag{82}$$

We regard the red index as optional: If it is missing, this is just the local ground state projector (apart from some prefactor, see Eq.(39)), otherwise, we place some excitation at the corresponding direct (dual) vertex. Now, we multiply all direct and dual ground state projectors in this form (where we might include the red index at some places), the dual ones coming first. The result can be written as:



, (83)

where we defined the tensor G as:



. (84)

For this we wrote the product of dual-dual (direct-direct) pairs as in the last step of Eq.(72).

Remark 25. This tensor network can be constructed for arbitrary lattices. The tensor G always has 4 pairs of virtual (and one pair of physical) indices, whereas the plaquettes that are formed by the pairs of indices can be arbitrary (in our example, it's tetragons). The orientation of the G tensors is always such that a pair of indices at the top or bottom (when we orient the letter G in the canonical way) of one G tensor is connected with the pair of indices at the left or right of another G tensor.

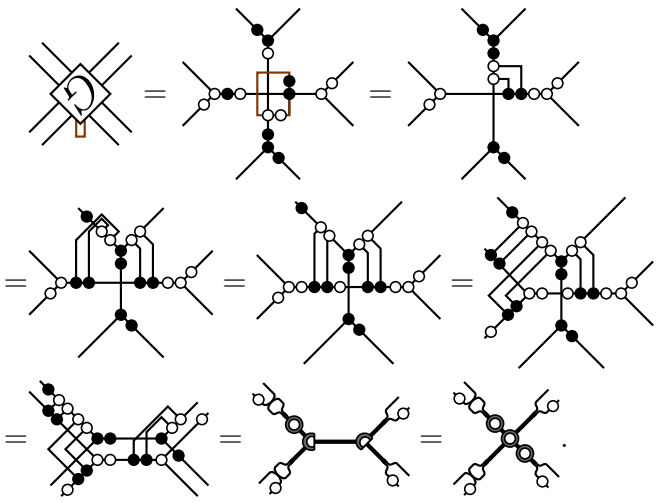
Remark 26. G tensors for other edge orientations that in Eq.(84) can as always be obtained by multiplying with the direct or dual 2-leg tensor on both physical indices. However, the orientations of the lattice can always be chosen such that it suffices to only rotate the different G-tensors.

The red dual (direct) symbols with additional open indices change the tensor network locally from a ground state projector to the direct (dual) representation

of the open-index algebra element at the corresponding direct (dual) vertex. If two such representations are on the same pair of indices, they form together a representation of the quantum double.

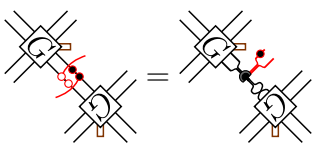
4.6 Degeneracy of ground state manifold and topological vector space

One fundamental property of the representation Eq.(83) is the multiplicity of its irreps, that in our case includes both the dimension of the topological vector space (for more excitations) and the additional degeneracy due to the lattice topology (that already exists for ground states). We get this multiplicity by constructing the corresponding irrep projector from the representation Eq.(83), tracing over the result, and dividing by the dimension of the irrep.
The corresponding trace of the tensor G that forms the tensor network turns out to be almost equal to the direct 4-leg tensor of the quantum double algebra:



(85)

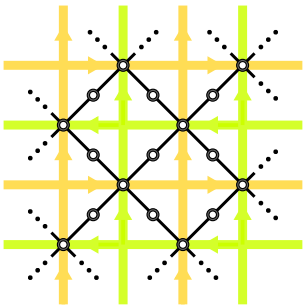
From steps 3 to 4 and 5 to 6, we used the pull-through equation for the mutual conjugation Eq.(15). From step 6 to 7, the associativity equation Eq.(6) for the dual tensors was used. In the end, we interpreted the tensor via the quantum double algebra symbols as expressed in Eq.(48:2) and Eq.(49). In between, 2-leg tensors were shifted around and used to flip the orientation of 3-leg tensors. When we now add some excitation to the site corresponding to the edge between two of the G-tensors, this is equivalent to a dual multiplication of the quantum double:



(86)

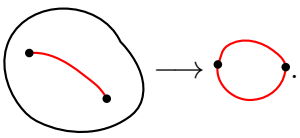
Unfortunately, the G tensor as in Eq.(85) has an additional direct 2-leg tensor at its ends, and the direct and dual part are interchanged. So we cannot match it with the dual quantum double tensors of Eq.(86) to work completely

at the level of the quantum double algebra. However, to examine the ground state degeneracy, the direct 2-leg tensors cancel everywhere, and we obtain the following tensor network:



(87)

We can now begin to contract Eq.(87) via fusion (Eq.(4)) and removal of plaquettes of the lattice defined by the contractions (Eq.(10:0,x)). For each such plaquette belonging to a direct (dual) vertex, we have thrown away a direct (dual) 0-leg algebra tensor in Eq.(82) and a dual (direct) 0-leg as in Eq.(39). So in total, we have to include the inverse of the dual 0-leg tensor of the quantum double for each plaquette, which exactly gets absorbed when we remove the plaquette by Eq.(10:0,x). What we end up with at the end by fusion of tensors and removal of plaquettes in Eq.(87) does not depend on the detailed structure of the lattice, but only on its topology.
Let's consider first closed surfaces: as we know how to contract Eq.(87) in the plain, we cut the surface along one or more lines till we get such a plain. All tensors in this plain reduce to one single direct tensor of the quantum double, which has one contraction with itself for every line we cut. For each point where lines end or meet, we didn't contract one plaquette of the network via Eq.(10:0,x), such that we haven't absorbed the corresponding normalization factor which is the inverse of the dual 0-leg tensor of the quantum double. So for the sphere, we have to cut along one line with two endpoints: (All closed surfaces will be shown as black contour from above, the cut lines will be colored, and the points where they meet or end are marked by black dots.)



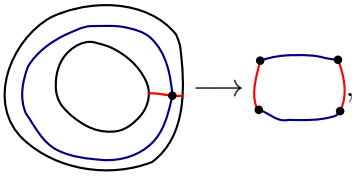
(88)

We have cut along one line, with two endpoints, so we end up with the following (where here and in the following all self-contractions of the direct tensor of the quantum double are understood to be dressed with the corresponding direct 2-leg tensor):

$$(\bullet)^{-2} = (\circ)(\bullet)^{-1} = 1. \tag{89}$$

So the ground state of the quantum doubles on a sphere is always non-degenerate. For a torus, we can cut it along two non-contractible loops that meet in one

point:



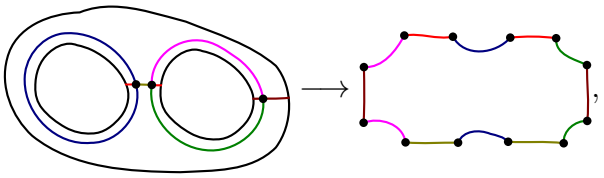
(90)

such that we get:

$$(\bullet)^{-1} = \mathrm{Tr}(\text{projector onto algebra center}) \\ = \dim(\text{center}) = \text{Nr. of irreps},$$

(91)

where we identified the result with the trace of the projector onto the direct center of the quantum double, as shown in Eq.(12) for general Hopf*-algebras. So on the torus we get one ground state for each irrep of the quantum double. For manifolds of higher genus we get similar expressions for the ground state degeneracy that however don't have such a clear interpretation anymore. For the double-torus, for example, one can cut along the following 6 lines, meeting at 3 points:

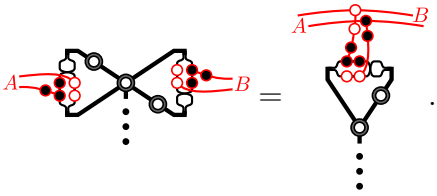


(92)

which yields:

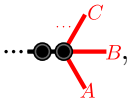
$(\bullet)^{-3}$. (93)

Contracting the tensor network with excitations is a bit more involved, but in principle doable. We only give a sketch here, without caring for normalization or additional 2-leg tensors: When we contract a plain as in Eq.(87) with some excitations (here shown for two excitations A and B), the only thing that remains is their total charge. So A and B get fused during the contraction:



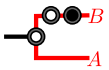
(94)

For topologically non-trivial surfaces (and non-abelian Hopf*-algebras), there will be one-particle excitations which have a multiplicity different from 0. So if we have multiple particles, their multiplicity degree of freedom consists of all different ways that they can fuse to each of those one-particle excitations. That is, the multiplicity for particles $A, B, C, \ldots$ looks like the following:

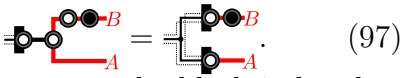


(95)

where the black dotted index is connected to the possible one-particle excitations. E.g. on the sphere, we would have to close this by the trivial irrep as there are no non-trivial one-particle excitations. However, all excitation configurations with non-trivial total charge can only be created by global operations, as braiding and re-fusing particles around non-contractible loops.
The knowledge of the topological degeneracy with excitations makes it possible to consider open boundaries as well: Such quantum double models are obtained by punching holes into closed surfaces by removing the corresponding representation. This removal means just ignoring possible excitations at those holes, such that the ground states of the surface with holes are just the states of the surface with holes that have arbitrary excitations at those holes and none elsewhere. In general, the multiplicity of a state with arbitrary excitations on a surface with holes is the number of distinct ways that those excitations can fuse together with any possible charge configuration on the holes and any possible single-particle configuration to the trivial charge. E.g. the disk (sphere with one missing direct vertex) has only one ground state, as there are no (i.e. only trivial) single-particle excitations on the original sphere. But arbitrary single-particle direct excitations are possible (with multiplicity 1), as the direct hole can absorb the corresponding counterpart.
A pair of excitations that was locally created fuses to the trivial charge, thus it has to be a particle-antiparticle pair. Each irrep α and its conjugate irrep fuse to the trivial charge in exactly one way, such that the multiplicity degree of freedom for each particle-antiparticle configuration has dimension 1, whereas the internal state is of dimension $\dim(\alpha)^2$. We can parametrise this space of two particles A and B fusing to the trivial charge by one algebra element in the following way:

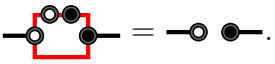


(96)



(97)

They fuse to the trivial charge for all particles we put to the black index due to Eq.(10:1,1):



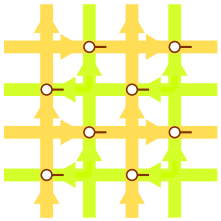
(98)

In fact it is very easy to see that Eq.(96) represents a particle-antiparticle pair: The direct 3-leg tensor does nothing but copying a particle, and the direct 2-leg tensor joined with the dual one maps between particle and anti-particle. In the form at the right of Eq.(97), one can see that the two internal indices on the left correspond to the internal state x of the irrep subspace.

4.7 Ground and excited states as tensor networks

With the representation Eq.(83) we can realize any map on the internal state x from Sec.(3.2) in any irrep at some places (combined with the ground state projector at most places). This includes projecting first onto a single (global) irrep α and mapping then (some or all internal states) to one fixed internal state. If we apply this map to an arbitrary initial state, we always end up

with a state in a specific irrep sector with a specific internal state. The initial state will decide in what invariant subspace corresponding to α we end up, i.e. what the state of the m-label from Sec.(3.2) we will have. The only thing we'll have to care about is to choose the initial state such that the result is non-zero. Let's begin with α equal to the trivial irrep sector (or ground state subspace). As initial state, we can take a state that is already invariant for all the dual representations, namely the tensor product of the direct 1-leg tensors at each spin:



(99)

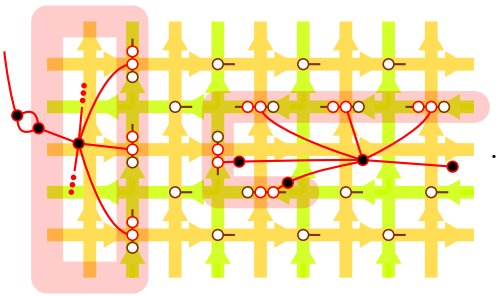
Now we want to include excitations. For simplicity, we will only consider a particle-antiparticle pair, i.e. a situation where α has two non-trivial components that are conjugate to each other:

$$\alpha = (\ldots, \epsilon, \beta, \epsilon, \ldots, \epsilon, \overline{\beta}, \epsilon, \ldots), \tag{100}$$

where ϵ denotes the trivial irrep and β is an irrep of the quantum double.

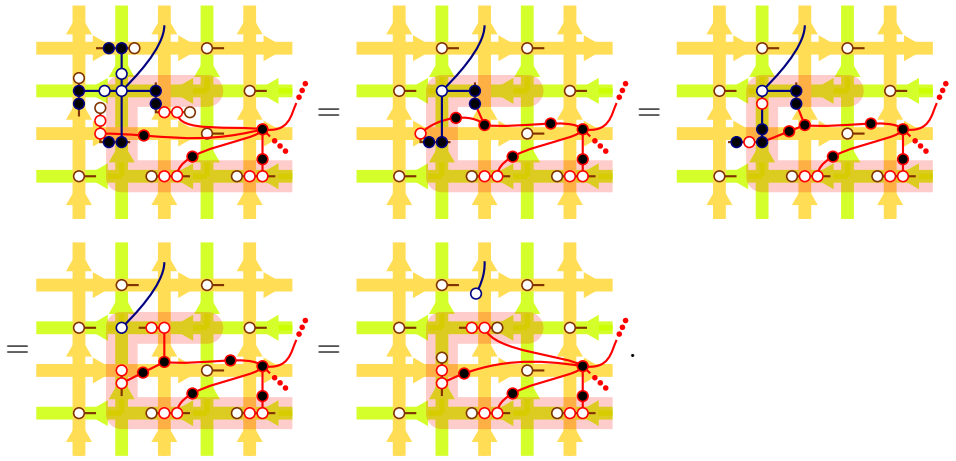
Remark 27. As we sketched in Sec.(4.6), this configuration has multiplicity 1 (times the ground state degeneracy on topologically non-trivial surfaces), so it is rather easy to analyze. Apart from that it can be created locally on top of a ground state in contrast to the single-particle excitations on non-trivial surfaces, for example.

The initial state Eq.(99) would yield 0 for all representations Eq.(83) that project onto α with a β, that is not a pure dual excitation. We get a good initial state by applying a direct string operator from Sec.(4.4) along some open dual path that has its endpoints just at the desired locations for the two excitations. As we want to be able to get any pair of excitations out of this string, we close it by the dual 1-leg symbol. Such an open string along some red shaded path is shown in the right half of the next tensor network. Instead of open strings, we can also act with the direct string operator along some closed dual loops on this state. Such an operator does not create any dual excitation for elements of the dual center, as there is no defined (coinciding) start- and endpoint. Such an closed string is shown in the left half of the tensor network:



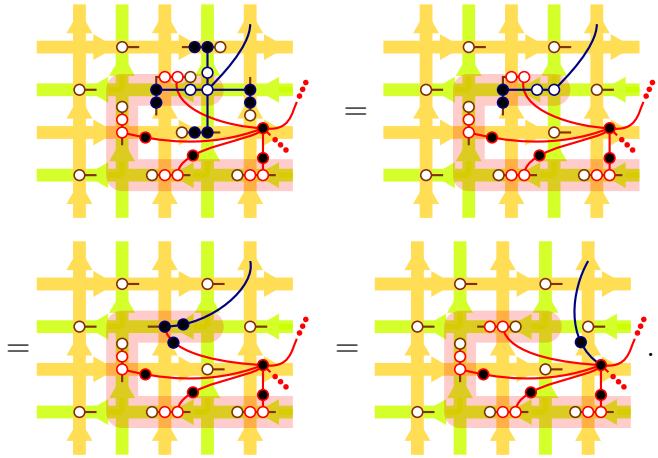
(101)

This tensor network state is already invariant for all the dual representations, except for those at the two endpoints. So all the dual ground state projectors (except for those at the two endpoints of the string) do not change the state. For representations without overlap with the string this is trivial, for those somewhere along the string, it is exemplarily shown here:



(102)

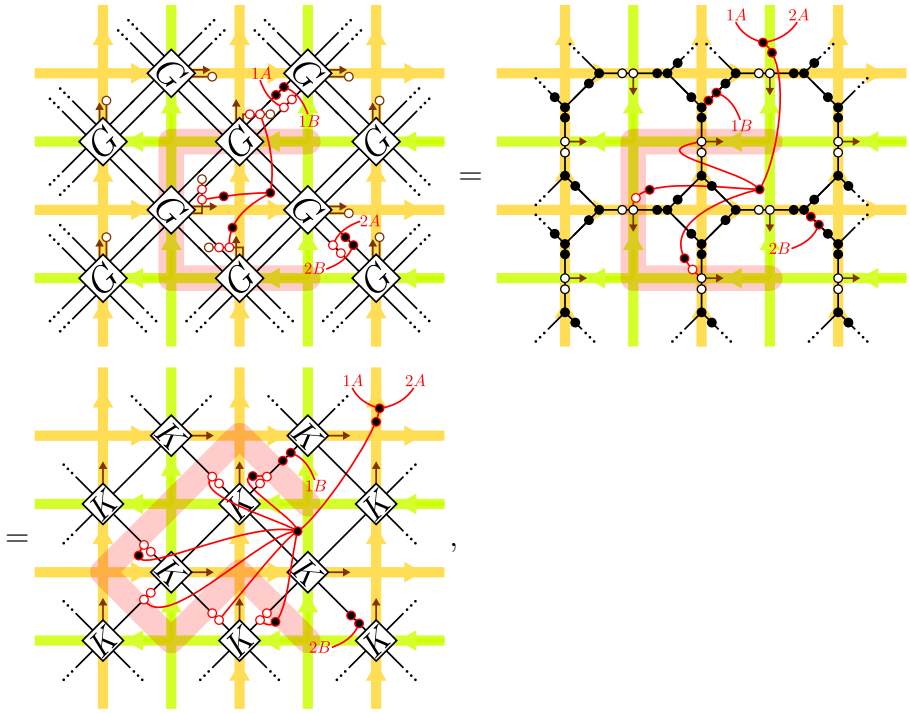
So there are no (i.e. only trivial) dual excitations along the string. In contrast, at the two endpoints, we have any kind of excitations:



(103)

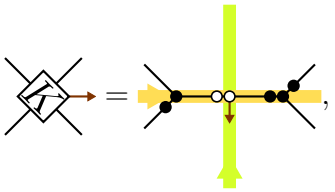
The tensor network Eq.(83) consists of two layers: It is the product of first all dual ground state projectors (eventually with some excitations) and then all direct ground state projectors. Because of Eq.(102) and Eq.(103), we can omit

the first layer and directly act with the second one on the state Eq.(101):



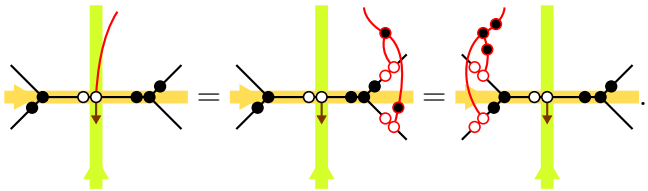
(104)

where we introduced the following tensor K:



(105)

and the fact that we can shift the red extensions corresponding to excitations on the virtual indices:

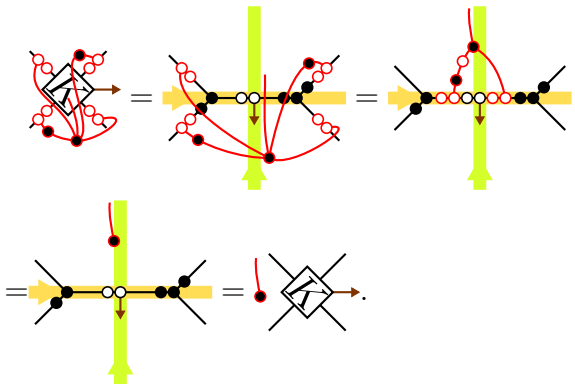


(106)

Note that this tensor is equivalent to the one used to build up the ground state tensor network in [2].

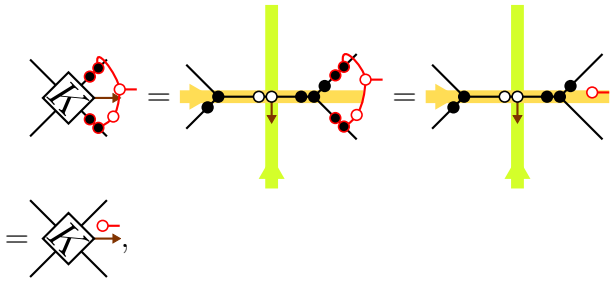
Remark 28. We also could have started with the tensor product of dual 1-leg tensors (instead of Eq.(99)) that is invariant under the direct representations and introduce dual strings in order to model excitations. We would get analogous results even if we don't change the order of dual and direct part in the definition of the quantum double. The tensor network representations we would obtain by this equal the ones we obtain here just by interchange of direct and dual tensors.

The tensor K has the three following symmetries on the virtual legs. First, a representation of the direct algebra:



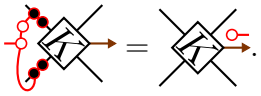
(107)

Then, a representation of the dual algebra on the right side:



(108)

and analogously a representation of the dual algebra on the left:



(109)

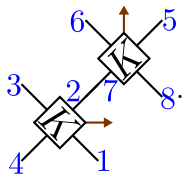
These are symmetries of some small quantum double model corresponding to the following lattice. The 4 virtual legs correspond to the 4 spins of the model at the 4 (direct or dual) edges:

(110)

where the 4 ends labeled by a correspond to the same direct vertex. The three symmetries correspond to the representations of the direct vertex c and the dual vertices b and d. The representation corresponding to a is also a symmetry of the tensor K, nevertheless it is equivalent to the one around vertex c. The other vertices e and f are not realized as symmetry (such that the surface corresponds to a miniature disk with a hole). If the tensor K is understood as a linear map between the virtual and physical degrees of freedom, it is an isometry (when we put the correct prefactor). The 4 virtual degrees of freedom are reduced to 1 physical degree of freedom via 3 different (direct and dual) symmetries.

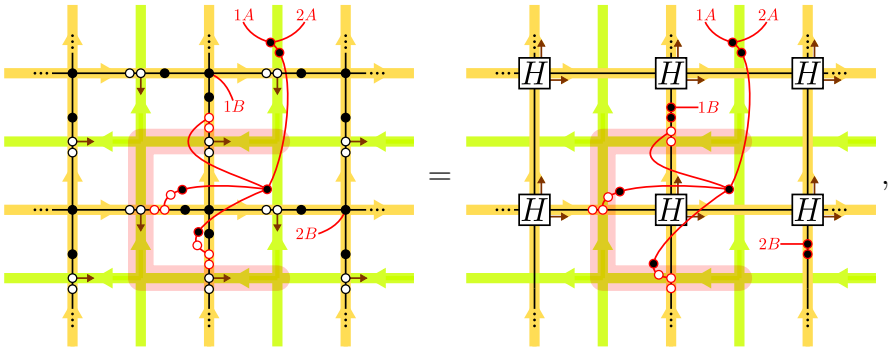
If we shift the excitation string as in Eq.(106), we get a new path, now with alternatingly direct and dual vertices as corners. Because of the direct symmetry

of the K-tensor Eq.(107), we can shift excitations from one side of the tensor to the others as for example also shown in Eq.(106). With these operations, we can deform the path connecting the two excitations arbitrarily. So all topologically identical paths are equivalent. Also note the behavior of the symmetries when we block 2 or several K-tensors, like the following:



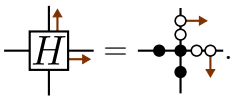
(111)

The direct symmetries of the single tensors around $(1,2,3,4)$ and $(5,6,7,8)$ will merge to a an analogous direct symmetry of the whole block around $(1,8,5,6,3,4)$. In general, when blocking arbitrary big regions of the tensor network Eq.(104), all direct symmetries will fuse to one big global direct symmetry around the whole blocked region. So this symmetry is stable under blocking which is a key feature of topologically ordered systems. This is not true for the two dual symmetries. When blocked, the dual symmetries $(3,4)$ and $(5,6)$ are unaffected, whereas the symmetries $(1,2)$ and $(7,8)$ are fused to a new dual symmetry on $(1,8)$. So the dual symmetries are always local and involve pairs of neighboring indices, which also holds for blocking of arbitrary big regions.
Starting from the second step in Eq.(104), by re-fusing the split-up dual tensors by Eq.(82) (or not splitting them up in first place), we can arrive at an alternative form of the tensor network:



(112)

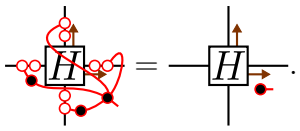
where we defined the tensor H as:



(113)

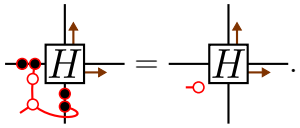
Remark 29. On arbitrary lattices, H is defined analogously with an arbitrary number of legs, one for each neighboring direct vertex. Every spin is attached to the virtual index of one of the two neighboring direct vertices, such that the "dressed" virtual legs of every H tensor are all next to each other.

The H-tensor has the following direct symmetry on the virtual legs:



(114)

and the dual symmetry:



(115)

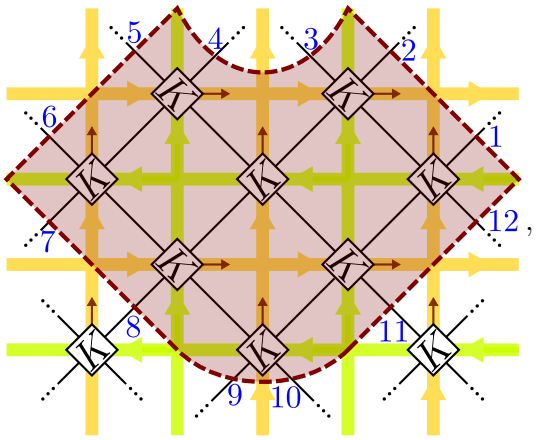
Those are the quantum double symmetries corresponding to the vertices b, e and c/a in the following lattice:

(116)

Remark 30. The tensor H as map between the virtual and physical indices is again an isometry. On general lattices, when H has k virtual legs, l of them attaches with a physical leg, one has 1 direct symmetry as in Eq.(114) and $(k-l)-1$ dual symmetries as in Eq.(115) between each pair of neighboring legs without physical indices. These together $k-l$ symmetries precisely reduce the k virtual to the l physical degrees of freedom.

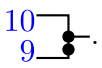
Again, the direct symmetry allows to move around the strings corresponding to excitations. It is also again stable under blocking of many tensors, whereas the dual symmetries stay local symmetries at the boundary.
So both tensors K and H that we found to generate tensor network representations of ground- and excited states had both a virtual symmetry corresponding to a miniature quantum double model. The direct and dual parts of that symmetry showed a qualitatively different behavior under blocking of tensors: Whereas the direct symmetry is stable under blocking and leads to a global symmetry around arbitrary big blocked pieces, the dual symmetries are always local on the boundary. This indicates that one can use the latter symmetries to reduce the effective dimension (or better, number) of virtual bonds by a local isometry on the virtual level. This is indeed the case: If we cut out some part of the K-tensor network given in Eq.(104) along a path that connects dual edges,

like the following



(117)

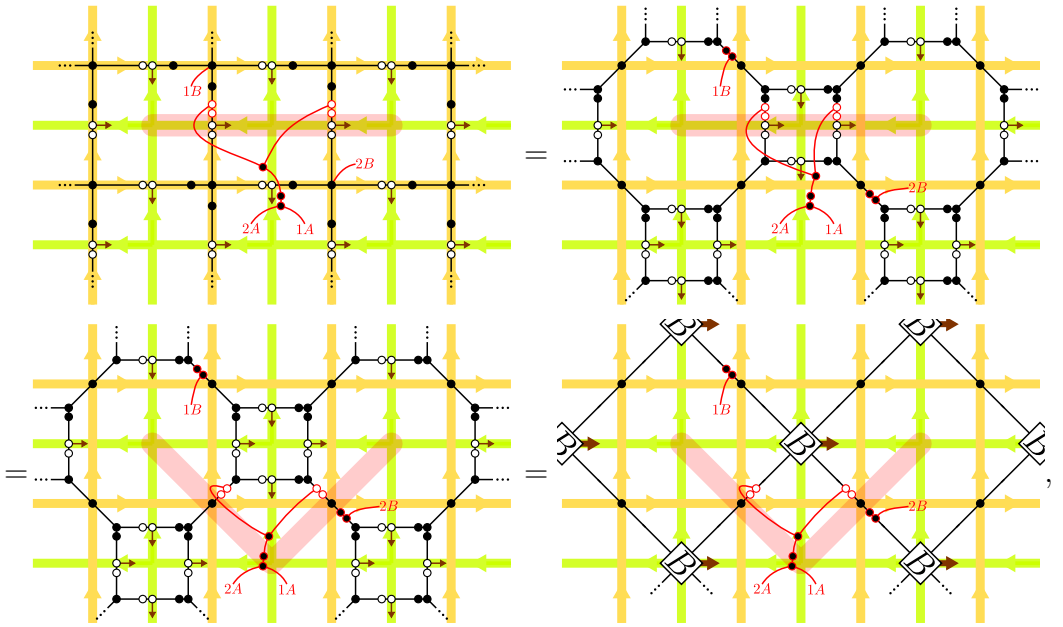
the cut boundary has dual symmetries acting on pairs of the indices that connect the cut region with the rest. As one can see from Eq.(108), in the example given, these are pairs of indices $1-2$, $3-4$, $5-6$, $7-8$, $9-10$ and $11-12$. So we know that those pairs of indices are always in the invariant subspace of the corresponding symmetry which has a lower dimension than the full virtual space. The symmetry is the tensor product of two regular representations of the dual of $\mathcal{G}$ which is isomorphic to G times the regular representation which has only a one-dimensional invariant subspace. Thus one can reduce the bond dimension from $G \times G$ to G. The corresponding isometry is (here exemplary shown for the index pair $9-10$, analogously for others):

$(\circ)^{-1}$ 

(118)

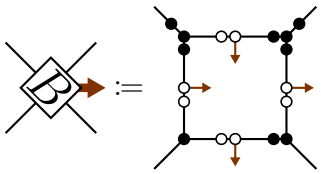
One can now cut the tensor network Eq.(104) into pieces as in Eq.(117) that each comprise one or more dual vertices. The result is particularly nice for bipartite lattices: There one can block all spins around every second (according to the bipartition) dual vertex. Every block is then connected to every neighboring one, and has as many virtual than physical legs. Equivalently, one can get the resulting tensor network from Eq.(112) by separating the dual tensor at the dual vertices by the fusion rules Eq.(4) such that the desired blocks are split from the rest. Thereby, the excitations can be shifted to the new virtual legs via the Hopf*-axioms Eq.(9:2,2) (the additional legs of the string that are

created thereby cancel by Eq.(10:1,1)):



(119)

with the tensor B defined as:



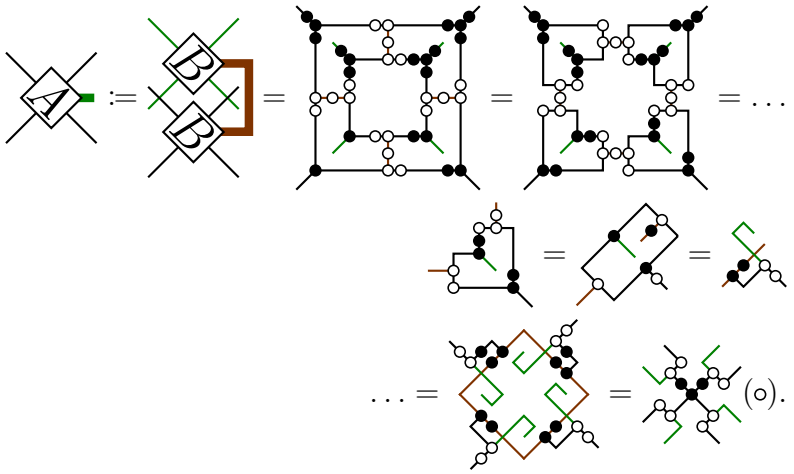
(120)

Remark 31. This form generalizes by adding more legs to arbitrary bipartite lattices, where each B-tensor corresponds to one dual vertex. For non-bipartite lattices, one would have to block more than one vertex sometimes which leads to a different form of the B-tensor, where we have more physical than virtual legs.

Remark 32. Note that when blocking the spins around some dual vertices, dual excitations corresponding to those vertices are not anymore representable as simple strings, but would imply a modification of the tensor at the endpoint of the string. We don't loose anything about the topological nature of the excitations if we simply forbid those excitations.

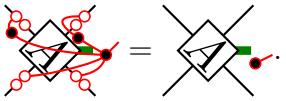
The invariance under the dual symmetry of the spins around some blocked dual vertex becomes a dual symmetry on the physical legs of the corresponding B-tensor. This means that the tensor B regarded as a map from the virtual to the physical legs is some linear map U followed by the projector A onto the subspace invariant under that symmetry. It turns out that B is in fact an isometry (apart from some factor), thus U can be chosen unitary. We can exchange the projection and the isometry ($B = U(U^{-1}AU)$) and obtain a corresponding symmetry acting on the virtual legs (letting $U^{-1}AU$ invariant).

As A differs from B only by the local physical unitary U, the ground states build from A or B are qualitatively equivalent. One obtains the projection A by $A = B^\dagger B$:



(121)

Both A and B are invariant under the same representation of the direct symmetry algebra $\mathcal{G}$ acting on the virtual legs:



(122)

This symmetry is just the one belonging to the quantum double for the following lattice:

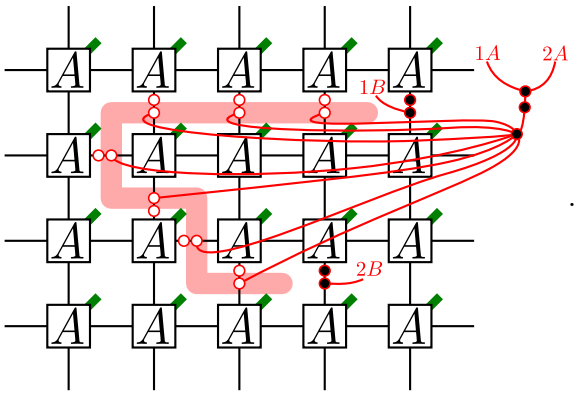
(123)

The symmetry Eq.(122) is implemented by the direct representation corresponding to c or equivalently a, the representations at b, d, e and f are not realized.

Remark 33. For group algebras, this is just the standard form of $\mathcal{G}$-isometric PEPS tensors as given in [1]. In general, A (and also the operators representing the symmetry) can be written as an MPO as shown in the penultimate step of Eq.(121), which are projector MPOs in the sense of [3].

Using the tensor A defined above, we can write down a fourth tensor network

representing the ground state with a pair of excitations:



. (124)

The red string belonging to an excitation can again be moved freely by the virtual symmetry of the A tensor Eq.(122).

Example 4.4: Consider as a special case $\mathcal{G}$ being a group algebra. Then the tensor B as a map from the virtual to the physical indices is in conventional notation:

$$B = \sum_{a,b,c,d \in G} |ad^{-1}, ab^{-1}, bc^{-1}, dc^{-1}\rangle \langle a, b, c, d| \,, \tag{125}$$

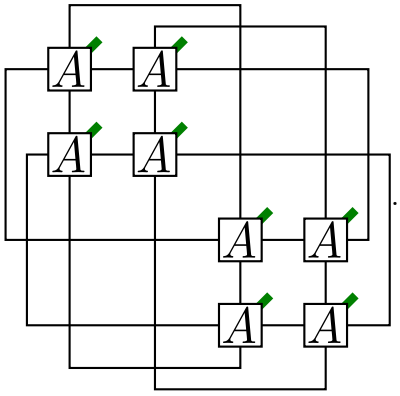
where we label the virtual (physical) indices is in counter-clockwise order beginning from the upper right (right) one in Eq.(121). The A operator is given by:

$$A = \sum_{g \in G} \sum_{a,b,c,d \in G} |ag^{-1}, bg^{-1}, cg^{-1}, dg^{-1}\rangle \langle a, b, c, d| \,. \tag{126}$$

Note that this is only the real part of the tensors (which are not complex-linear operators). In order to express the complex conjugation in this language, one would have to write some spins as ket and some as bra.

This last tensor network representation is more efficient than the others: As we have gotten rid of the dual virtual symmetries that are local at the virtual boundary, as illustrated in Eq.(117), it effectively contains a lower number of virtual bonds. This makes it more suitable for numerical algorithms.
Note here that the information about the whole system is already contained in the (no matter which) tensor network representation of its ground state: For example, we can construct from the tensor network a Hamiltonian, the so-called *parent Hamiltonian* [1] in the following way: One groups together patches of multiple of the PEPS-tensors and regards them as map from the virtual indices on the boundary to the physical indices at the bulk. If the patches are large enough, this map will not have full range. So we can build the projector onto the supported subspace as shown here exemplary for a 2×2 patch in the A-tensor

representation:



(127)

Those local projectors sum up to a global Hamiltonian for the ground state. Also, in the same way in which we deduced the virtual symmetries from the global ones above, we can also get the physical local symmetries from the symmetries of the PEPS-tensors.

4.8 Parametrising the degeneracy due to the lattice topology

In the preceding section, we derived different tensor network representations of the ground state or states with a pair of (locally created, i.e. trivially fusing) excitations on some local section of the lattice. Now, let's look at the situation for the full manifold. If it is topologically non-trivial, we get an additional degeneracy for the ground state and its excitations, as we've seen in Sec.(4.6). It turns out that we can parametrise this additional degree of freedom by also considering closed loops in the initial state Eq.(101) which lead to virtual strings along closed loops in our tensor network representations. If the loops are topologically trivial, we can contract them to a point and make them disappear by the direct symmetry of the PEPS tensors, so those do not yield new states. But if they are non-contractible, like the horizontally and vertically winding loops on a torus, there's no possibility to remove them. Nevertheless, we can still deform topologically identical loops into each other or fuse them together to one new loop. If we want to parametrise the topological degeneracy for arbitrary manifolds, we have to generalize this a bit: We can consider arbitrary strings of different $\mathcal{G}$-elements that have two ends each (so they also have to be directed) and meet each other at those points. At each such meeting point, all connected lines have to multiply to identity in order not to cause a dual excitation. Now insert such a network of strings along paths that cut the surface to an open plane, as illustrated in Eqn.(88),(90),(92) and attach one $\mathcal{G}$-element to each (differently colored) line, such that they fulfill the restriction of all intersection points (black dots). Not all of these configurations yield different states: one can insert a (topologically trivial) string in the plain and bring it to the boundary to fuse it to all lines (that are each twice at the boundary of the plain). This will simultaneously conjugate the $\mathcal{G}$-elements corresponding to all of the

lines.

Proposition 7. All ground states of the quantum doubles can be parametrised in the following way: Cut the surface along lines such that it becomes an open plain. Attach a $\mathcal{G}$-element to each cut line, such that the $\mathcal{G}$-elements multiply to the direct unit on every point they meet. Those configurations of $\mathcal{G}$-elements label the ground states up to simultaneous conjugation of all $\mathcal{G}$-elements.

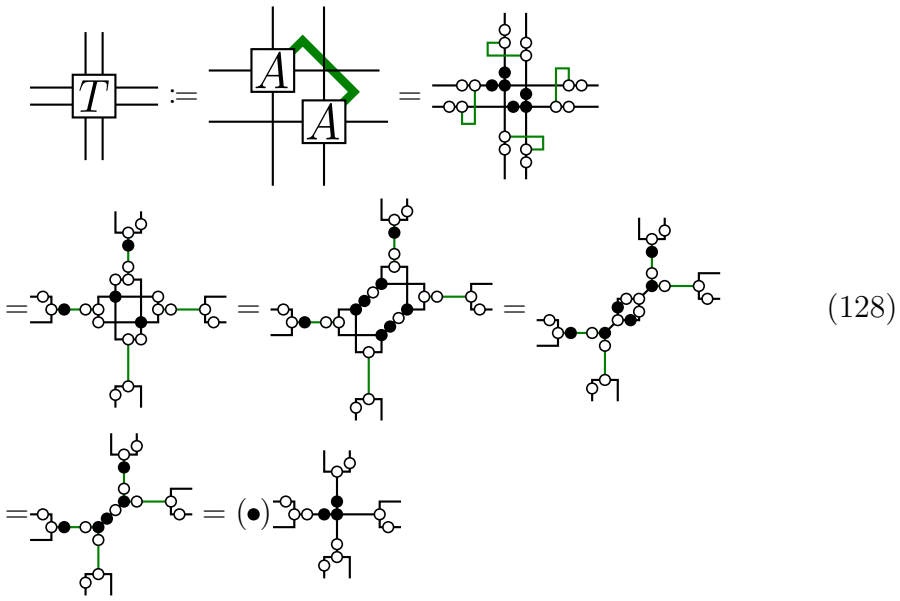
Remark 34. If our string network is bigger, such that it divides the plain into multiple plaquettes, we can insert loops in every plaquette and fuse them with the plaquette boundary. Those (direct) plaquette gauges together with the (dual) intersection constraints are nothing but the symmetries of a quantum double model. So what we did in fact in the construction above (with only one big plaquette) is to reduce the size of our quantum double model to a miniature model with the same topology that only has one direct vertex, which is represents both endpoints of every direct edge. The action of this single direct vertex is nothing but the simultaneous conjugation (because it applies to both ends of every direct edge) of all $\mathcal{G}$-elements. For each intersection point there is one dual vertex, such that the intersection constraints are the ground state projectors of the corresponding dual symmetry.

Example 4.5: Consider a toric lattice and a group algebra $\mathcal{G}$. The torus can be cut to a plain by two lines g and h intersecting at one point. The intersection constraint $ghg^{-1}h^{-1} = 1$ dictates that g and h commute. Thus the ground states are spanned by commuting pairs, up to simultaneous conjugation. This conjugation gauge can be fixed by first fixing g_C so some specific group element in each conjugacy class C. Then one can still simultaneously conjugate by any group element in the normaliser $N[g_C]$ without destroying this first gauge fixing. This remaining gauge has to be fixed by h that itself is element of $N[g_C]$. This can be done by fixing h to some fixed element h_D for each conjugacy class D of $N[g_C]$. So ground states are labeled by conjugacy classes C together with conjugacy classes D of the respective normaliser, or equivalently by C and irreps α of the normaliser. This fits to our result in Eq.(91) that there is one ground state for each irrep of the quantum double, together with the characterization of those irreps in App.(A).

4.9 The transfer operator

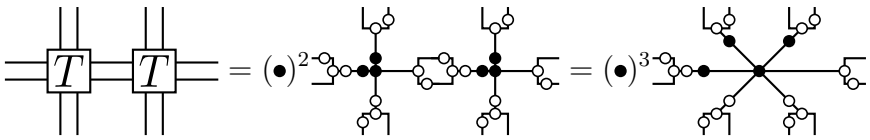
Statistical physics is all about computing expectation values, so the overlap of a state with an observable. Such an observable might be (the projector onto) another state. Let's consider the case where we want to calculate the overlap between two states of the quantum double that are distinct from the ground state only at a few places, in one of the tensor network forms developed in Sec.(4.7). Let's choose the MPO-isometric form in terms of A-tensors. The

network arising from the contraction is generated at most places by the following 8-leg tensor T:



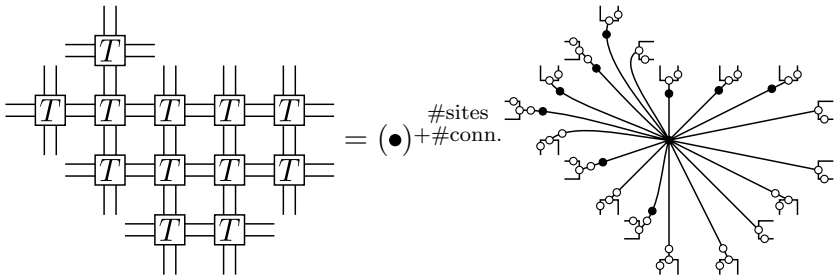
(128)

In fact, the resulting network will completely constitute of T-tensors with additional strings on the links and projectors at the string ends at both the ket and bra layer. When there's no such string or projector between two neighboring T-tensors, we can combine them to one bigger T-tensor with the same structure, just with more links:



(129)

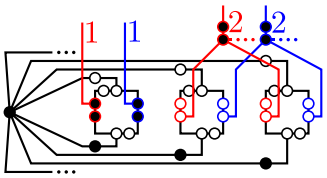
In the same way, we can also block together several T-tensors to arbitrary big T-tensors corresponding to some connected region:



(130)

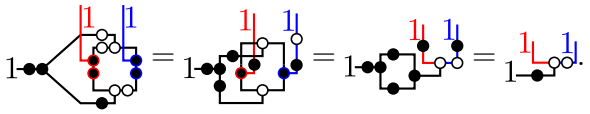
Remark 35. With the help of the T-tensor, we can directly calculate the overlap between different excitations. Of course, we know by construction that they are all orthogonal, but it's instructive to do the calculation. So lets consider two states with two different excitations at the same point. For the calculation it's easier to imagine an infinite plain and to move the respective second particle to infinity. This has the effect that the internal state of the second particle

becomes a multiplicity degree of freedom for the single-particle configuration. The corresponding state tensor network has a half-infinite string. Now we move the strings of the two states we want to contract to the same path, and contract the T-tensor as in Eq.(130) for the whole region around this path. We will get the following remaining network to contract (where the two strings are in blue and red, the T-tensor is in black; we will not care for prefactors):



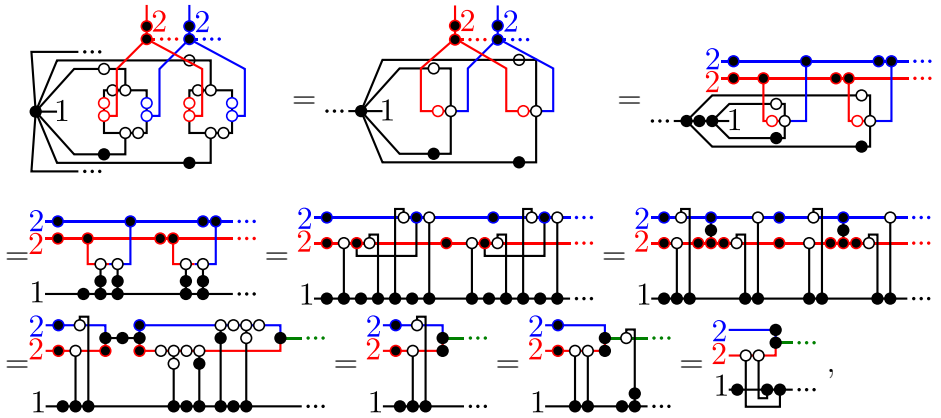
. (131)

Let's consider first the contraction at the endpoint of the string:



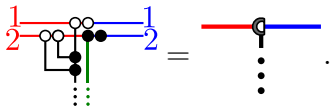
(132)

The rest of the string simplifies in the following way (the black label 1 indicates how this rest is connected to the endpoint above):



(133)

where the lines with the dotted ends only represent the connection of the string-end with the rest, such that any operation acting only on them can be absorbed. In step 7, we see that it doesn't matter how long the string is.
Combining those two parts, the whole overlap reduces to:

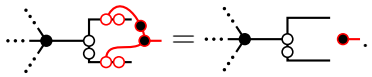


. (134)

This is nothing but the right-regular representation of the quantum double Eq.(49), transforming the first (blue) excitation into the second one (red). The index that transforms them comes from the boundary condition at infinity. It splits up into two parts: The black part comes from the transfer operator, thus from the virtual bonds of the ground state tensor network. The green part

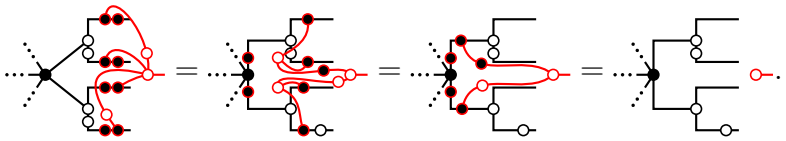
comes from the virtual string between our excitations and their anti-particles at infinity. This illustrates why we actually need virtual strings to model particles: In our ground state tensor networks, we got rid of one layer of representations corresponding to the dual part of the quantum double. That's why the ground state tensor network cannot transport the information about the dual part of the second excitation anymore, and this has to be done via strings.
The right-regular leaves the irrep and the internal state of the particle invariant, but changes the multiplicity degree of freedom, which is nothing but the internal degree of freedom of the second particle at infinity. This means that the two one-particle states are already orthogonal if their irrep or the local degree of freedom of the particle are different, the rest depends on the boundary condition, i.e. the local degree of freedom of the (virtual) particle at infinity.

Let us now examine the symmetries of the T-tensor. First, there is the direct symmetry at the virtual level of the A-tensor Eq.(122) that immediately carries over two one of those symmetries at each the ket and bra layer of the T-tensor. Second, there is a local direct symmetry for each single ket/bra index pair:



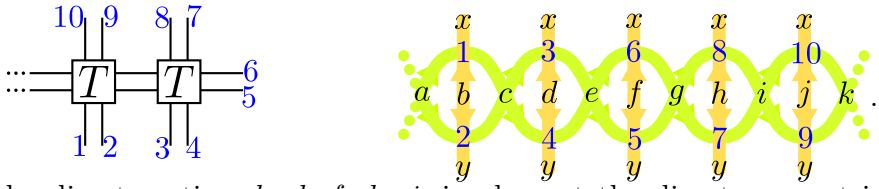
(135)

Third, there is a local dual symmetry that comprises all neighbouring pairs of ket/bra index pairs:



(136)

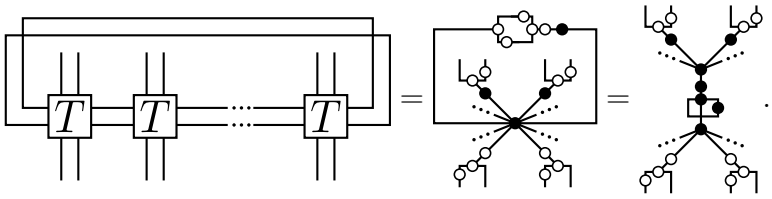
Those representations are precisely the symmetries of the (effectively one-dimensional) quantum double model on the following lattice:



(137)

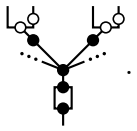
Here the direct vertices b, d, f, h, j, implement the direct symmetries corresponding to the index pairs $1-2$, $3-4$, $5-6$, $7-8$, $9-10$. The dual vertices a, c, e, g, i, k correspond to the dual symmetries on the indices $(\ldots, 1-2)$, $(1-2, 3-4)$, $(3-4, 5-6)$, $(5-6, 7-8)$, $(7-8, 9-10)$, $(9-10, \ldots)$. The line ends labelled by x (y) correspond all to the same direct vertex that correspond to the global direct symmetry inherited from the A-tensor Eq.(122) on the ket (bra) layer, that is stable under blocking. The orientation-dependent additional direct and dual 2-leg symbols and the order of ket and bra indices in the T-tensor that can be seen from Eq.(130) are just such that the lattice structure Eq.(137) is the same everywhere around the blocked region.
One can also build the T-tensor for a topologically non-trivial region on the

surface, as for example a disk with a hole or a non-contractable ribbon around a torus. Then the region has two (or more) non-connected (one-dimensional) surfaces, and the structure is a little bit different. The following shows the situation for a surface with one hole:



(138)

So here the different one-dimensional surfaces (here at the top and bottom, respectively) are separated by a projection onto the dual center. This tensor interpreted as an operator $\mathbb{T}$ between the two sides of the ribbon is called the *transfer operator* [17]. $\mathbb{T}$ is a partial isometry, i.e. its singular values are either 0 or 1, as is easy to see: Eq.(138) is already in SVD form. The (better, one choice of those with singular value 1) singular vectors are given by:



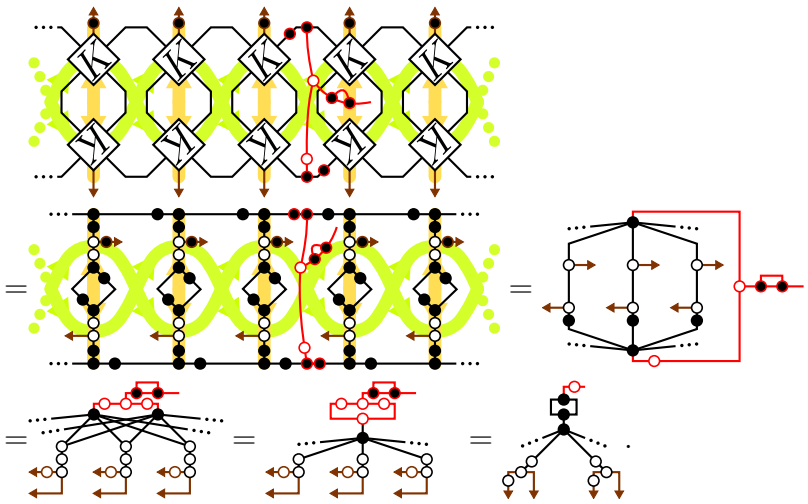
(139)

Those singular vectors have the same structure as the transfer operator itself, apart from the additional label corresponding to an element in the dual center. They share all local symmetries of the transfer operator, but the two global symmetries are lost: they now map between the different singular vectors.

Remark 36. For abelian direct algebras, the simultanous action of x and y is still a symmetry, and for non-abelian direct algebras, one could extract a generalised normaliser subalgebra for which the simultanous action is a symmetry. However, this symmetry seems to be rather unnatural, as for example it is qualitatively different if we interchange dual and direct algebra in our description.

So the symmetries of the singular vectors correspond to the lattice in Eq.(137), without the representations of the direct vertices x and y. We can now consider the quantum double model living at the boundary for this lattice: The topology of the corresponding surface is a disk with a hole (or open cylinder), the missing two vertices corresponding to one open boundary each. The singular vectors Eq.(139) are exactly the ground states for this model. They can be parametrised by the tensor network Eq.(104) with two (conjugate) direct

excitations sitting at the vertices x and y that we ignored:



(140)

From step 2 to 3, we fused the dual tensors around each direct vertex, then we applied Eq.(9:2,x). The ground state without any charges at the ignored vertices x and y has the same structure as the transfer operator itself.
One can also create real excitations at the local direct/dual vertices of the boundary system that are not ignored. Those excitations can themselves be modeled as strings on the virtual legs of the ground state tensor network of the boundary system as in Eq.(104). Also, they can be created by ribbon operators on the physical level of the boundary system (which is the virtual level of the bulk). One can also consider the excitation strings in the bulk as operators on the virtual level. We conjecture that those virtual operators of the bulk system are exactly the physical operators of the boundary system that create a pair of excitations there. The ribbons corresponding to these creation operators are a bit artificial: Let's assume, we want to create a particle-antiparticle pair at the two sites $a-b$ and $j-k$ in the lattice Eq.(137). The dual path of the ribbon is then formed by $a-c-e-g-i-k$ and involves the spins $1-3-6-8-10$ (alternatively $2-4-5-7-9$), whereas the parallel direct path is formed by $b-x-j$ (alternatively $b-y-j$) and involves the spins $1-10$ (alternatively $2-9$). Of course, those two different alternatives for the form of the ribbon yield the same state when acting to the ground state of the boundary system. The form of those ribbons fit to the form of the virtual strings in the bulk: they consist of direct representations all along the dual path in the boundary system, and only two dual representations at the two endpoints corresponding to the direct path in the boundary system.

4.10 Deformation and symmetry breaking

We can also build the T-tensor for arbitrary (deformed) tensor network states. The local symmetries will get lost under the deformation (they will get smeared out somehow), but the global symmetries x and y corresponding to the ket and

bra layer individually will remain. Of course, the corresponding transfer operator will not be an isometry any more, so instead of singular vectors we will have to consider its fixed points. At this point it's conceptually easier to assume the transfer operator build for some straight column in an square lattice on a torus or an infinite plain, such that we can identify its ingoing and outcoming space. The x and y symmetries again map between different fixed points. The fixed points can be numerically calculated in the form of iMPS (tensor network states at the boundary) by successively applying the transfer operator in MPO-form and truncating the bond dimension via SVD or similar. Then one can create a particle in the boundary system with the creation operator of the RG fixed-point system and calculate its overlap with the ground state (and other particles). In the boundary system, this is nothing but testing the corresponding string order parameter. In the bulk, the corresponding contraction is equal to calculating the overlap of a deformed excitation state with the deformed ground state (or another deformed excitation state). So the order is different in both cases: In the bulk, one first builds excitations and then looks if their deformation is still qualitatively (i.e. they are independent particles when far away from each other) a good excitation, whereas at the boundary, one first deforms the ground states and looks if the excitations created by the undeformed (string-) operators are qualitatively the right ones.
As an example of symmetry breaking in our model, let's consider a group algebra $\mathcal{G}$ with a normal subgroup $\mathcal{H}$, which is a subalgebra in the sense of Eq.(68). As a deformation take the projection P_H onto $\mathcal{H}$ on every spin independently. The result is nothing but embedding the quantum double corresponding to $\mathcal{H}$ into the original one. The deformed model is again a fixed-point model with the following local symmetries:

- The restriction of the direct representations to the subgroup $\mathcal{H}$ as in Eq.(66).
- The full dual representations.
- The (right- or left-) regular representation of the dual of $\mathcal{G}$ on every single spin, restricted to the dual of the quotient $\mathcal{G}/\mathcal{H}$. This is nothing but a projector algebra onto the different cosets of $\mathcal{H}$.

Remark 37. As $\mathcal{H}$ is normal, the symmetry algebra $\mathcal{G}$ (and also its direct representation) factorises into $\mathcal{H}$ and its quotient algebra $\mathcal{G}/\mathcal{H}$. The last one of our deformed symmetries above is just the operator algebra that creates the excitations corresponding to this quotient subsymmetry.

We can now restrict our physical space to the subgroup $\mathcal{H}$ for all individual spins. This means that we simply forbid excitations corresponding to the last symmetry above. We can then also restrict our dual representations to the subgroup $\mathcal{H}$. Our remaining symmetry corresponds to $D(\mathcal{H})$ and is a true subrepresentation of the old representation of $D(\mathcal{G}$ in the sense of Eq.(67), such

that we can observe how the old excitations split up and condense to the new excitations or how they are confined.

Example 4.6: Consider the breaking of symmetry from the quantum double $D(S_3)$ to the subalgebra $D(Z_3)$ obtained after a restriction to the cyclic elements in both components. The irreducible representations are given in Eq.(145). As all irreps of $D(Z_3)$ are one-dimensional, the higher-dimensional irreps of $D(S_3)$ have to split up or be confined. This splitting up is shown in Eq.(145) in the second row of the box corresponding to each irrep: All irreps with negative-sign elements in the second slot are confined, whereas all others split up in the following way:

$$\begin{aligned}
&(C_0^{S_3}, I_0^{S_3}) \Rightarrow (C_0^{Z_3}, I_0^{Z_3}) \\
&(C_0^{S_3}, I_1^{S_3}) \Rightarrow (C_0^{Z_3}, I_0^{Z_3}) \\
&(C_0^{S_3}, I_2^{S_3}) \Rightarrow (C_0^{Z_3}, I_1^{Z_3}), (C_0^{Z_3}, I_2^{Z_3}) \\
&(C_1^{S_3}, I_0^{Z_2}) \Rightarrow \text{confined} \\
&(C_1^{S_3}, I_1^{Z_2}) \Rightarrow \text{confined} \\
&(C_2^{S_3}, I_0^{Z_3}) \Rightarrow (C_1^{Z_3}, I_0^{Z_3}), (C_2^{Z_3}, I_0^{Z_3}) \\
&(C_2^{S_3}, I_1^{Z_3}) \Rightarrow (C_1^{Z_3}, I_1^{Z_3}), (C_2^{Z_3}, I_1^{Z_3}) \\
&(C_2^{S_3}, I_2^{Z_3}) \Rightarrow (C_1^{Z_3}, I_2^{Z_3}), (C_2^{Z_3}, I_2^{Z_3})
\end{aligned} \tag{141}$$

One can also see that the first two irreps are condensed.

5 Summary and outlook

We were examining symmetries and excitations in the quantum double models via the language of tensor networks. After introducing all necessary algebraic structures in Sec.(2), we defined excitations in general systems as (irreducible) violations of a local symmetry in Sec.(3). We saw that a sufficient condition to get independent, orthogonal excitations is that all symmetries are isometric and commute among each other. When defining the direct and dual representations in the quantum double in Sec.(4.1), we saw that we have to relax this condition a bit there: Those symmetries do not commute among each other in general, but the representations of their dual centers do. Because of this, the direct and dual representations form together a representation of a bi-crossed product which is just the quantum double (Sec.(4.2)). So we can have independent violations of the single representations if they are in the dual center, and independent violations of the quantum double symmetry of a neighbouring pair of direct and dual representations (containing the former ones, Sec.(4.3)). In Sec.(4.4), we saw that those particles can only be created in pairs leading to a string form of the corresponding creation operators. Then we built a tensor network that is the mixture between a ground state projector in most places and local representations of the symmetry at some places in Sec.(4.5). In Sec.(4.6) we used the trace of this tensor network to derive algebraic expressions for the ground state degeneracy on arbitrary surfaces. We also outlined how the topological degeneracy looks like for arbitrary particle configurations and open surfaces. We also (Sec.(4.7) constructed tensor networks corresponding to a state with a pair of excitations by acting with the projector tensor network on a corresponding initial state. Thereby we got rid of one of the two layers coming from the direct and dual part of the double, respectively. This comes with the price of getting a virtual string (corresponding to the lost layer) between the two particles that mediates second part of the information shared by them. We looked at different blockings of this tensor network and at the symmetries of the corresponding PEPS-tensors. We found that all of them have symmetries acting on the virtual legs only, that correspond to some miniature quantum double model. The corresponding direct symmetries merge to a global symmetry on the boundary after blocking of multiple PEPS-tensors, whereas the dual symmetries always stay local. The former ones are also those that allow to move around the virtual strings between excitations. We also found a blocking that completely gets rid of the latter dual symmetries and thus has an effectively lower bond dimension. In Sec.(4.8) we considered closed (loops or networks of) virtual strings and saw that they can be used to parametrise the topological degree of freedom of ground (and excited) states. Then (4.9) we investigated the T-tensor that arises from contracting the overlap between two low-lying excited states in tensor network form. We found that this tensor living on the virtual boundary (ket and bra layer) of the system has local and global symmetries of a (effectively one-dimensional) quantum double model. The T-operator on a ring-like shape can be interpreted as an operator between the inside and

outside boundary. The (non-zero) singular vectors of this operator are ground states with respect the the local symmetries of the boundary quantum double. The virtual strings in the bulk can be interpreted as the ribbon operators for this boundary quantum double that create excitations there. So we saw that excitations in the boundary system are in one-to-one correspondance with excitations in the bulk. Finally, we considered deforming our quantum double to a quantum double of a subgroup and had a look at charge condensation and confinement from a symmetry perspective.
There are still a few things that could be improved or added. For example we didn't derive explicitly the form of the ribbon operators in Sec.(4.4). Also, our method for calculating topological degeneracies in Sec.(4.6) is maybe not the most elegant one: One could instead imagine a procedure that transforms the trace tensor network into the same for the lattice with one edge removed, and thereby fuses the excitations that are on the disappearing contractions. Further, there was no time in the end to explicitly verify our conjecture about the boundary system. This should be possible in a straight-forward manner by testing the quantum double symmetry of the boundary system at the endpoints of the virtual strings.
There are also few fundamental questions remaining open. A clear picture why and which models with local, (almost) commuting symmetries are important is still missing. To this end, it would be nice to have a clear understanding of the relation between phases of matter, renormalisation group flows and their fixed-points, and local commuting symmetries. Also it is not clear in general, what property of the local symmetries is responsible for the fact that particles are created in pairs, i.e. that the corresponding creation operators have string-like form. To get an idea on this, it would be helpful to also apply our Hopf*-algebraic approach to other fixed-point models with topological order, like the twisted quantum doubles, or string-net models.

A Diagonalising the quantum double algebra of a group algebra

In this appendix, we describe how to find the diagonalising transformation of any quantum double emerging from a group algebra, assuming the knowledge of the conjugacy classes and the irreps of their normalisers for this group.
Eq.(49) is the right-regular representation of the quantum double if we regard the right index pair as the incoming, the top left index pair as outcoming operator indices, and the bottom left index pair as the algebra element that is represented. We can decompose the according representation space into invariant subspaces to find the irreps and the diagonalising transformation. At this point, it is more convenient to switch from the graphical tensor-network notation to index notation with Einstein summation convention. Then, we will denote the right-regular representation in Eq.(49) as $(R_{g,h})^{i,j}_{i',j'}$, where i', j' are

the right (when looking from inside to outside) and left parts of the outgoing index, i, j is the incoming index pair, and g, h corresponds to the represented algebra element. Then the entries of R are either 0 or 1, and the non-zero entries are:

$$(R_{g,h})^{i,j}_{i',j'} = 1 \quad \Longleftrightarrow \quad (i', j') = (ig^{-1}, gjg^{-1}) \wedge h = j \tag{142}$$

So the action of the algebra element g, h onto the basis vector i, j is to project onto the space where $h = j$ and then to map

$$i, j \longrightarrow A_g(i, j) \coloneqq (ig^{-1}, gjg^{-1}).$$

We can now find the irreps of the Drinfield double by searching for invariant subspaces of this right-regular representation.

Remark 38. One could equally take the left-regular representation to do this. The right-regular form has the advantage, that the action is separated into 2 parts which only depend either on g or on h: A permutation in the standard basis with A_g, and a projection onto the element h of the standard basis in the second component.

All invariant subspaces we find will be spanned by subsubspaces that are a product of anything in the first component and a standard basis element in the second component. Thus, they will automatically be invariant under the projectors in the second component. Those invariant spaces can be found and refined in 3 steps:

1. For any conjugacy class C of G, the subspace

   $$V^C \coloneqq \mathrm{Span}\{(i, j) | i \in G, j \in C\}$$

   is invariant, as the only effect of A_g on the second component is just a conjugation. The dimension of V^C is $|G||C|$.

2. The variable $X(i, j) \coloneqq iji^{-1}$ is invariant under the action of A_g:

   $$X(A_g(i, j)) = (ig^{-1})(gjg^{-1})(ig^{-1})^{-1} = iji^{-1} = X(i, j)$$

   Thus, within V^C, for any $x \in C$, the subspace

   $$V^{C,x} \coloneqq \mathrm{Span}\{(i, j) | X(i, j) = x\}$$

   is invariant. Now consider a fixed $x \in G$ and i, $j \in G$ such that $X(i, j) = x$. Then for any $i' \in G$, $X(i', j) = x$ if and only if there is $n \in N[x]$, such that $i' = ni$. Thus $\{i | X(i, j) = x\}$ forms exactly one right coset of $N[x]$. ($N[x]$ denotes the normaliser of x, the set of group elements commuting with x.) In particular, there's one unique such coset $\mathrm{Co}_x(j)$

for each $j \in C$. For any $i \in G$, define $J_x(i) \in C$ such that $i \in \mathrm{Co}_x(J_x(i))$. Then

$$V^{C,x} = \mathrm{Span}\{(J_x(i), i) \coloneqq D_i^{C,x} | i \in G\} \tag{143}$$

expresses the invariant subspace in a more constructive form. We choose one arbitrary element $\mathrm{co}_x(j)$ from each coset $\mathrm{Co}_x(j)$ (1 for $\mathrm{Co}_x(x) = N[x]$) and define $N_x(i) \in N[x]$ for every $i \in G$ such that $i = N_x(i)\,\mathrm{co}_x(J_x(i))$.

3. The action of A_g onto the spanning vectors of $V^{C,x}$ labelled by $i \in G$ as in Eq.(143) is equal to the right-regular representation of G.

$$A_g(J_x(i), i) = (J_x(ig^{-1}), ig^{-1}).$$

Now decompose $\mathrm{Span}\{(J_x(n), n) | n \in N[x]\}$ into $\dim(\alpha)$ invariant (under the right-regular representation) subspaces $V_x^{C,x,\alpha,m}$ for any irrep α of $N[x]$ with multiplicity label m. For all $j \in C$ define the subspace

$$V_j^{C,x,\alpha,m} \coloneqq A_{\mathrm{co}_x(j)}(V_x^{C,x,\alpha,m}) \subset \mathrm{Span}\{(i, j) | i \in \mathrm{Co}_x(j)\}.$$

The direct sum of all those subspaces is invariant under the action of the full algebra:

$$V^{C,x,\alpha,m} \coloneqq \bigoplus_j V_j^{C,x,\alpha,m}.$$

To see that it is invariant, consider the action of A_g on the direct sum component $V_j^{C,x,\alpha,m}$, after decomposing g as

$$g = \underbrace{g\,\mathrm{co}_x(j)\,\mathrm{co}_x(J_x(\mathrm{co}_x(j)^{-1}g^{-1}))}_{a}\underbrace{\mathrm{co}_x(J_x(\mathrm{co}_x(j)^{-1}g^{-1}))^{-1}}_{b}\underbrace{\mathrm{co}_x(j)^{-1}}_{c},$$

such that $A_g = A_a \circ A_b \circ A_c$: First, A_c maps $V_j^{C,x,\alpha,m}$ to $V_x^{C,x,\alpha,m}$ by definition. Then, A_b maps this to $V^{C,x,\alpha,m}_{J_x(\mathrm{co}_x(j)^{-1}g^{-1})}$. In the end, a is of the form $a = d\,\mathrm{co}_x(J_x(d^{-1})) \in N[x]$, so A_a lets the last subspace invariant by definition.

Those subspaces are irreducible: The dual part of the action (projection onto h) can be used to project onto any of the subspaces $V_h^{C,x,\alpha,m}$, and each of those subspaces is by construction already irreducible under the subset $\{A_g | g \in N[x]\}$ of the direct part of the action. The indices C and α together in $V^{C,x,\alpha,m}$ label the different kinds of irreps with dimension $|C|\dim(\alpha)$, whereas x and m label the $|C|\dim(\alpha)$ different invariant subspaces for the same kind of irrep. As for any regular representation, the multiplicity of each irrep equals exactly its dimension.

Irreps with $C = 1$ are referred to as *pure direct excitations* (or vertex or charge excitations or chargeons), whereas irreps with $\alpha = 1$ are called *pure dual excitations* (or plaquette or magnetic excitations or fluxons).

A similar construction is also thinkable for the quantum doubles of general Hopf*-algebras, one would have to replace "group element" by "dual irrep", "conjugacy class" by "set of dual irreps invariant under direct conjugation", and so on.

B Diagonalisation of $D(S_3)$

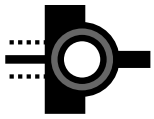
As example for the construction in App.(A) we consider the quantum double of the full symmetric group of 3 elements, S_3. We will denote its elements in cycle notation:

$$\{(), (12), (13), (23), (123), (132)\}.$$

The standard basis element of $D(S_3)$ corresponding to e.g. the group elements (12) and (132) will be denoted as $(12|132)$. S_3 has 3 conjugacy classes: The identity class, $C_0^{S_3} := \{()\}$, the transposition class $C_1^{S_3} := \{(12), (13), (23)\}$, and the full cycle class $C_2^{S_3} := \{(123), (132)\}$. In addition to that, we need the irreps of all normaliser, S_3 for $C_0^{S_3}$, Z_2 for $C_1^{S_3}$ and Z_3 for $C_2^{S_3}$. S_3 has 3 irreps: The trivial irrep $I_0^{S_3}$, the sign irrep $I_1^{S_3}$, and the standard irrep $I_2^{S_3}$. The other two normaliser are commutative groups G, for which we will denote the irrep with $\omega = e^{\frac{2\pi i n}{|G|}}$ by I_n^G. So the irreps of $D(S_3)$ can be identified with:

$$\begin{aligned}&(C_0^{S_3}, I_0^{S_3}), (C_0^{S_3}, I_1^{S_3}), (C_0^{S_3}, I_2^{S_3}), (C_1^{S_3}, I_0^{Z_2}), (C_1^{S_3}, I_1^{Z_2}),\\ &(C_2^{S_3}, I_0^{Z_3}), (C_2^{S_3}, I_1^{Z_3}), (C_2^{S_3}, I_2^{Z_3}).\end{aligned} \tag{144}$$

Applying the construction from App.(A), we get for the diagonalising transformation:



=

$(C_0^{S_3}, I_0^{S_3})$
$(C_0^{Z_3}, I_0^{Z_3})$
$(\|) + (12\|) + (13\|) + (23\|) + (123\|) + (132\|)$

$(C_0^{S_3}, I_1^{S_3})$
$(C_0^{Z_3}, I_0^{Z_3})$
$(\|) - (12\|) - (13\|) - (23\|) + (123\|) + (132\|)$

$(C_0^{S_3}, I_2^{S_3})$	
$(C_0^{Z_3}, I_1^{Z_3})$	$(C_0^{Z_3}, I_2^{Z_3})$
$(\|) + e^{\frac{1}{3}2\pi i}(123\|) + e^{\frac{2}{3}2\pi i}(132\|)$	$(12\|) + e^{\frac{1}{3}2\pi i}(23\|) + e^{\frac{2}{3}2\pi i}(13\|)$
$(12\|) + e^{\frac{2}{3}2\pi i}(23\|) + e^{\frac{1}{3}2\pi i}(13\|)$	$(\|) + e^{\frac{2}{3}2\pi i}(123\|) + e^{\frac{1}{3}2\pi i}(132\|)$

$(C_1^{S_3}, I_0^{S_3})$		
confined		
$(\|12) + (12\|12)$	$(123\|23) + (13\|23)$	$(132\|13) + (23\|13)$
$(132\|12) + (13\|12)$	$(\|23) + (23\|23)$	$(123\|13) + (12\|13)$
$(123\|12) + (23\|12)$	$(132\|23) + (12\|23)$	$(\|13) + (13\|13)$

$(C_1^{S_3}, I_1^{S_3})$		
confined		
$(\|12) - (12\|12)$	$(123\|23) - (13\|23)$	$(132\|13) - (23\|13)$
$(132\|12) - (13\|12)$	$(\|23) - (23\|23)$	$(123\|13) - (12\|13)$
$(123\|12) - (23\|12)$	$(132\|23) - (12\|23)$	$(\|13) - (13\|13)$

$(C_2^{S_3}, I_0^{S_3})$	
$(C_1^{Z_3}, I_0^{Z_3})$	$(C_2^{Z_3}, I_0^{Z_3})$
$(\|123) + (123\|123) + (132\|123)$	$(12\|132) + (23\|132) + (13\|132)$
$(12\|123) + (13\|123) + (23\|123)$	$(\|132) + (132\|132) + (123\|132)$

$(C_2^{S_3}, I_1^{S_3})$	
$(C_1^{Z_3}, I_1^{Z_3})$	$(C_2^{Z_3}, I_1^{Z_3})$
$(\|123) + e^{\frac{1}{3}2\pi i}(123\|123) + e^{\frac{2}{3}2\pi i}(132\|123)$	$(12\|132) + e^{\frac{1}{3}2\pi i}(23\|132) + e^{\frac{2}{3}2\pi i}(13\|132)$
$(12\|123) + e^{\frac{1}{3}2\pi i}(13\|123) + e^{\frac{2}{3}2\pi i}(23\|123)$	$(\|132) + e^{\frac{1}{3}2\pi i}(132\|132) + e^{\frac{2}{3}2\pi i}(123\|132)$

$(C_2^{S_3}, I_2^{S_3})$	
$(C_1^{Z_3}, I_2^{Z_3})$	$(C_2^{Z_3}, I_2^{Z_3})$
$(\|123) + e^{\frac{2}{3}2\pi i}(123\|123) + e^{\frac{1}{3}2\pi i}(132\|123)$	$(12\|132) + e^{\frac{2}{3}2\pi i}(23\|132) + e^{\frac{1}{3}2\pi i}(13\|132)$
$(12\|123) + e^{\frac{2}{3}2\pi i}(13\|123) + e^{\frac{1}{3}2\pi i}(23\|123)$	$(\|132) + e^{\frac{2}{3}2\pi i}(132\|132) + e^{\frac{1}{3}2\pi i}(123\|132)$

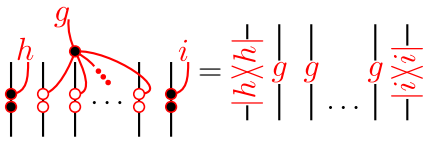
. (145)

Here, each irrep index corresponds to one block (see top row of each block), the left- and right internal states are the matrix row and column. The right index of the standard basis is written to the matrix entries in sparse form. The second top-row of the block will become important later for the study of symmetry breaking. The blocks for the first three $D(S_3)$-irreps correspond to the S_3-irreps in the first component with the second component fixed to the identity, as this is the only element in that conjugacy class. For all other $D(S_3)$-irreps the normaliser irreps are one-dimensional, but the conjugacy classes have more than one element. So the different entries all have the same normaliser-irrep superposition of elements inside of one coset in the first component, but different cosets in the first component and different conjugacy class elements in the second component. Note again that those coefficients are only determined up to similarity transforms on each single matrix. The subspaces of spanned by the rows of one $D(S_3)$-irrep-matrix are invariant under the left-regular action of $D(S_3)$, whereas the column subspaces are invariant under the right-regular representation. For any numbers i, j, there is an algebra element which maps column i to column j by its left-regular action and row i to row j by its right-regular representation. For the presented $D(S_3)$-irrep matrices corresponding to $C_1^{S_3}$, $(123) \in D(S_3)$ maps between rows/columns 1/2 and $(132) \in D(S_3)$ maps between rows/columns 1/3. For the $D(S_3)$-irrep matrices corresponding to $C_2^{S_3}$, the element mapping between rows/columns 1/2 is $12 \in D(S_3)$.

C Instruction for building the virtual strings

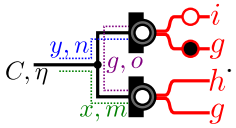
We have given the form of the virtual strings in Eq.(104) with two open pairs of red indices corresponding to two quantum double excitations. However, as those two particles always have to fuse to the trivial charge, the string is already parametrised by one irrep label and two internal states, as shown in Eq.(97). In this appendix, we will describe in detail what the strings parametrised like this look like for a group algebra, i.e. we will explicitly combine the raw form of the strings Eq.(104) with the parameterisation in Eq.(97) and the diagonalising

transformation of the quantum double of a group algebra in App.(A).
Let's first write the virtual strings in a more conventional form (which can be done for a group algebra only)



$$= |h\rangle\langle h| \otimes U_g \otimes U_g \otimes \ldots \otimes U_g \otimes |i\rangle\langle i| =: |g, h, i\rangle\,, \tag{146}$$

and also label the parameterisation Eq.(97) accordingly:



(147)

The label C, η are a conjugacy class of $\mathcal{G}$ and an irrep of the corresponding normaliser that together form an irrep of the quantum double of $\mathcal{G}$. All internal indices (x, m), (y, n), (g, o) split into two parts, where the first part is a group element $x, y, g \in C$, and the second part is an internal index of η. The labels x, m and y, n together form the internal state of the excitation, that was labeled x in Sec.(3). From App.(A), we know immediately that first part of the right internal state g, o (namely g) equals the right algebra-index of the diagonalising transformation (which equals the string label). So it's consistent that the right internal indices as well as the string labels have to be the same for both excitations. This string label is summed over the whole conjugacy class C. The labels x and y determine the right cosets of which normaliser (namely $N[x]$ and $N[y]$) we are restricted to for the labels h and i. For each g, the corresponding right coset is the one with elements c such that $cgc^{-1} = x$ (for h) or $cgc^{-1} = y$ (for i). The corresponding superposition for the h-label (i-label) is the superposition of η, m, o (η, n, o) in $N[x]$ ($N[y]$), shifted to the correct coset by multiplication from the right with an element of the coset. This corresponds to the following step-by-step instruction:

1. Choose a conjugacy class C, determine its normaliser $N[C]$ (isomorphic for all elements of C) and choose one irrep η of $N[C]$. C together with η determine the particle species.

2. Get (and choose the gauge of) the Fourier transform of $N[C]$ for the block corresponding to η, so a superposition of group elements c for every pair of internal states.

3. Choose two elements $x, y \in C$, and two internal states $m, n = 1, \ldots, \dim(\eta)$ of η. x, m and y, n are the internal state corresponding to each of the excitations, respectively.

4. From the Fourier block, take the two columns corresponding to the left internal states m and n. Let $A_o(c)$ and $B_o(c)$ denote the weight of the

group element c in these superpositions for the internal states m, o and n, o, respectively.

5. For each $g \in C$, choose and determine one element $a_g \in G$ ($b_g \in G$), such that $a_g g a_g^{-1} = x$ ($b_g g b_g^{-1} = y$), and shift the superposition I by a (and aditionally invert it for b):

$$A_o^g(c) \coloneqq A_o(c a_g^{-1}), \qquad B_o^g(c) \coloneqq B_o(b_g c^{-1}),$$

and write $|A_o^g\rangle$ and $|B_o^g\rangle$ for the corresponding superpositions.

6. The virtual string corresponding to the excitation pair with species C, η and internal state $(x, m), (y, n)$ is given by:

$$\begin{aligned} &\sum_{g \in C, o=1,\dots \dim(\eta)} \sum_{i,j} A_{m,o}^g(h) B_{n,o}^g(i) \, |g, h, i\rangle \\ &= \sum_{g \in C, o=1,\dots \dim(\eta)} |g\rangle \otimes |A_o^g\rangle \otimes |B_o^g\rangle \, . \end{aligned} \tag{148}$$

Note that, in contrast to abelian groups, the string and its two endpoints do not factorize into a product state with respect to g, h and i. (There the superposition is a product between a single element g and two character superpositions for h and i.) Instead, they are mutually entangled in a relatively complicated way. When compared to the abelien case, the computation suffers from an additional summation over g and o.

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Selbstständigkeitserklärung

Hiermit versichere ich, dass ich diese Masterarbeit zum Thema "Symmetries and excitations in the quantum double models - a tensor network approach" selbstständig verfasst habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, sowie Zitate kenntlich gemacht.

Mir ist bekannt, dass Zuwiderhandlung auch nachträglich zur Aberkennung des Abschlusses führen kann.

München, den 20. September 2016	
Ort, Datum	Unterschrift